

Timing of Auctions of Real Options*

Lin William Cong[†]

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Abstract

This paper endogenizes auction timing and initiation in auctions of real options. Because bidders have information rent, a seller faces a “virtual strike price” higher than the actual exercise cost. She inefficiently delays the auction to encourage bidder participation and utilizes the irreversible nature of time to gain partial control over option exercises. The seller’s private benefit at option exercise may restore efficient auction timing, but option exercises are always inefficiently late. When she lacks commitment to auction timing, bidders always initiate in equilibrium, resulting in earlier option exercise and higher welfare than auctions proscribing bidder initiation. Overall, auction timing modifies the distribution of the bidder valuations and has important impact on bidding strategies, auction design, and real outcomes.

JEL Classification: D21; D23; D44; D82; L26

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[†]University of Chicago Booth School of Business. Email: Will.Cong@ChicagoBooth.edu; Tel: +1 (773) 834-1436; Address: 5807 S Woodlawn Ave, Chicago, IL 60637.

1 Introduction

Auctions of real options are prevalent in licensing and patent acquisitions, leasing of natural resources, real estate development, M&A deals, venture capital and private equity markets, and privatization of large national enterprises.¹ Classical auction studies and the extensive literature applying auction theory in corporate finance (Hansen, 2001; Dasgupta and Hansen, 2007) treat the timing of auctions as exogenous. Recent studies on dynamic auctions and auction-like activities such as corporate mergers allow endogenous timing of the sale (e.g., Lambrecht, 2004; Morellec and Zhdanov, 2005; Gorbenko and Malenko, 2016), but typically ignore post-auction option exercises that determine the real option values in the first place.

To understand the impact of auction timing on the sales of real options, I combine the real options framework with that of auctions. I find that because a seller has to pay information rent to bidders, she has divergent preferences over the option exercise. In formal auctions, she inefficiently delays the auction to gain partial control over option exercises and encourage bidder participation. When a seller lacks commitment to auction timing and design, bidders always initiate the sale in equilibrium, accelerate option exercises, and can improve welfare because they utilize private signals to time the auction. The paper is the first to underscore that the irreversible nature of time renders auction timing and real option exercise interdependent. It contributes to auction theory by demonstrating how auction timing and initiation affect the competitive bidding and real outcomes, with welfare implications and relevance for policy and real life practice, especially in corporate finance settings in which assets are typically embedded with real options and agents strategically time the sales.

Specifically, a risk-neutral seller owns a real investment option but lacks the expertise to best exercise it. She therefore sells it to potential risk-neutral bidders privately informed about their heterogeneous option-exercise cost. Bidders all pay an initial cost at the transfer of ownership, and generate, upon exercising the option, a cash flow whose process follows a

¹Bolton, Roland, Vickers, and Burda (1992) describe the privatization policies in Central and Eastern Europe. Pakes (1986) and Schwartz (2004) discuss patents as real options. In corporate finance, many decisions entail optimal timing of taking certain actions such as shutting down a plant, going public, adopting a new technology, and launching a new product. For example, post M&A deals, the firms have to decide when to integrate certain units or to make new investments viable only after the merger (e.g., if they require pooling patents).

geometric Brownian motion. Agents interact in continuous time in three sequential stages. In the first stage, the seller (or potentially a bidder once we allow bidder initiation) strategically initiates the auction. In the second stage, participating bidders bid and the seller allocates the asset in exchange for the winner's payment. In the final stage, the winning bidder optimally times the exercise of the option.

I first examine formal auctions in which the seller commits to auction design, which includes both auction timing and auction rules. I use a mechanism design approach to show that because of the information rent the seller has to pay, she effectively receives the payoff from an option with a virtual exercise cost higher than that the winning bidder faces. At a first glance, the seller is powerless over the option exercise because, after all, the winning bidder owns the real option and determines its exercise. But the irreversible nature of time allows her to hold up the bidders — the bidder can exercise the option only after getting the ownership through the auction, but not before — which forces the bidder to partially incorporate the seller's preference on option exercise. The seller therefore can partially control the option exercise through auction timing.

On the one hand, delaying an auction means the seller has to delay receiving the revenue. On the other hand, delaying an auction can push the bidders' options more in-the-money on average, which increases bidder competition, encourages bidder participation (the optimal reserve price and bidder participation also increase when the auction is held at higher cash flow level), and delays bidders' investment cost at the auction (or any social cost associated with holding an auction). The overall effect is a modification in the distribution of participating bidders' valuation normalized to time-zero value. The seller balances these tradeoffs and holds the auction later than a social planner would, precisely because the higher virtual exercise cost she faces.

Can a seller always commit to auction timing and rules? The government certainly can in oil-lease auctions, wireless spectrum auctions, or privatization of state-owned enterprises. But many other sales to competitive buyers feature sellers lacking such commitment; that is, bidders can instead approach the seller to trigger an auction-like process with potential adjustments of offers, as seen in corporate takeovers where bidders decide what to offer and often can initiate the contact or negotiation.² In such informal auctions, prior literature

²One prominent M&A case involves Microsoft's \$8.5 billion cash acquisition of the voice-over-IP service Skype—its largest acquisition to date—for the portfolio of real options such as Windows phone integration.

shows that she effectively holds an English auction to allocate the asset as long as it generates positive net payoff. I therefore model the auction timing and initiation game by allowing bidders to approach the seller with an offer, after which the seller can time an English auction to sell the item.

In informal auctions, I find that bidders always initiate in equilibrium regardless of the divergence in preference for option exercise between the seller and bidders. To see this, note that the seller times the auction to maximize the option value of the second-best bidder, while a bidder times the auction to maximize the information rent, neither of which necessarily maximizes social welfare. Both the seller and bidders dynamically update their beliefs about the type of bidders present based on (the absence of) initiation, yet the seller's option value starts to be eroded later than that of the bidders' because the seller receives the second best real option, which becomes "in the money" after the best real option does.

Consequently, option exercises are earlier than those in formal auctions timed by the seller. Bidder initiation is also more socially efficient, achieving the first-best in terms of social welfare when bidders cannot force an auction upon initiation or ownership transfer is frictionless. The intuition is that bidders have private information on the option exercise cost, which is useful in timing the sale so that the option value does not inefficiently erode.

The main results and intuition in the paper are robust to introducing seller's private benefit or cost in option exercise, interdependent values, contingent bids, etc..³ Even though security bids alter the bidders' option exercises, the seller can still use auction timing to partially align bidders' option exercise timing with her preference, if the bidders tend to exercise the option later than what she prefers. When the seller lacks commitment to auction timing and security design, the bidders still always initiate because in equilibrium their bids are all cash-like (as shown in Cong, 2017), rendering the initiation game the same as in cash auctions.

Google's largest acquisition with \$12.5 billion for Motorola Mobility Company in 2012 also gave it the option to develop the portfolio of patents Motorola held. Both deals were bidder-initiated and against the backdrop of potential rival bids, and were effectively auctions. Some licensing agreements and contracts in the entertainment industry also fall in these categories.

³In reality, a seller may have intrinsic preferences for the project's future that differ from the bidders. The inventor of a technology or a product might develop emotional attachment (Tjan (2011) and Matyszczyk (2015)), prefer an earlier commercialization, or favor later product shutdown; the manager of a firm may not fully internalize the reputation cost of the entrepreneur who originally founded the company, but instead care more about firm performances in her relatively short tenure. I model this by introducing an additional payoff to the seller upon option exercise.

Moreover, as long as the seller is not too biased towards early exercise, she delays the auction beyond what is socially optimal. When she strongly prefers early exercise, however, she loses the partial control because of the uni-directional flow of time: she may accelerate the auction, but the winning bidder would still wait to exercise the option. Auction timing can only delay but never accelerate option exercise, relative to the standard real options problem.

Literature — This paper is foremost related to the literature on auction theory (e.g., Krishna, 2009), especially its applications to model corporate finance transactions, such as mergers and acquisitions and sales of scarce resources (e.g., Bulow, Huang, and Klemperer, 1999; Boone and Mulherin, 2007; Povel and Singh, 2010). Extant papers typically focus on the auction design and outcomes when an asset is already up for sale. Notable exceptions include an early study by Wang (1993) that considers auction timing as a seller’s strategic decision. A more recent study by Chaves and Ichihashi (2018) finds that sellers typically hold auctions inefficiently late with stochastic bidder arrival and departure.

An emerging theoretical literature on the initiation of M&A deals examines the timing of competitive sales. Hackbarth and Miao (2012) analyze the timing of mergers in a real options framework. Gorbenko and Malenko (2017) connect bidders’ financial constraints to their incentives to initiate deals, bids, and payment methods. Gorbenko and Malenko (2016) endogenize initiation of first-price auctions with changing valuations. Also closely related is Chen and Wang (2015), which considers two-sided private information and endogenous initiation in one-on-one M&A negotiations.

Although my model applies to the settings they consider, my focus is on selling real options with post-auction exercise. Because of this distinction, the irreversibility of time endows the seller partial control over the option exercise, making auction timing a strategic decision. Moreover, complementary to Gorbenko and Malenko (2017, 2016), bidder initiation in informal auctions involves second-price and English auctions and is driven by Bayesian learning of bidder distribution instead of financial constraints in first-price auctions.

An emerging literature also examines the irreversibility of time in dynamic corporate decisions, and depending on the direction of preference divergence between seller and buyer, draws rather asymmetric conclusions. Grenadier, Malenko, and Malenko (2016) examine the scope for and the structure of delegation and communication in a dynamic environment

with an uninformed principal and an informed but biased agent. Guo (2016) studies the same topic by considering the optimal mechanism without transfers when the agent prefers to experiment longer than the principal. Instead of communications or contracting between principals and agents, this paper focuses on another prevalent economic activity: auction – the transfer of ownership and control to competitive buyers with endogenous entry and time-varying valuations. Moreover, while the direction of the conflict of interest (timing) still remains important, it interacts with competitive bidding and bidder’s information rent, and the asymmetric impact no longer manifests itself based on a simple dichotomy of the direction of bias.⁴

Finally, my paper complements the emerging literature on agency issues in a real-options framework broadly. For example, Grenadier and Wang (2005), Cong (2012), and Gryglewicz and Hartman-Glaser (2015) study distortion of investment incentives due to adverse selection and moral hazard. Grenadier and Malenko (2011) and Morellec and Schürhoff (2011) examine signaling through option exercises. Most closely related is Cong (2017), which analyzes the post-auction moral hazard in auctions of real options with security bids, taking the time of the auction as exogenous. I contribute by endogenizing auction timing and initiation and demonstrating for the first time how they influence option exercises.

The remainder of the paper proceeds as follows. Section 2 describes the economic environment and sets up the model. Section 3 derives optimal option exercise, bidding equilibria, and endogenous auction timing. Section 4 allows bidder initiation and characterizes timing for informal auctions. Section 5 discusses extensions such as seller’s private bias in option exercise, interdependent values, and bidding with securities. Section 6 concludes. The appendix contains all the proofs and extended discussion.

⁴ In this regard, closely related is Malenko and Tsoy (2017) which studies auction design when buyers rely on biased experts. The authors show that while revenue equivalence theorem holds in static mechanisms, advisors communicate information gradually, resulting in more efficient allocation and higher or lower revenues depending on the direction of expert bias. These results derive from the irreversibility of running prices in Dutch or English auctions, which is similar to the irreversibility of time. That said, the paper does not consider endogenous auction timing, and focuses on transmission of non-verifiable information, rather than the post-auction option exercise that I examine.

2 Auction Environment

A risk-neutral revenue-maximizing seller with discount rate $r > 0$ owns a project with an embedded option. We can think of the seller as a technology-oriented entrepreneur and the option as the product's commercialization or the start-up's sale to other companies, investors in the public market, or business managers. We can also think of the seller as governments selling natural resources such as oil tracts for private firms to exploit later.

The state variable that summarizes the project's potential profit is public and evolves stochastically according to a geometric Brownian motion (GBM):

$$dX_t = \mu X_t dt + \sigma X_t dB_t, \quad (1)$$

where B_t is a standard Brownian motion under the equivalent martingale measure, μ is the instantaneous conditional expected percentage change per unit time in X_t , and σ is the instantaneous conditional standard deviation per unit time. Note X_t could represent the present value of a stream of future cash flows the project can generate. I assume $\mu < r$ to ensure a finite value of the option.

The seller does not have the expertise to exploit the option but can auction the project to N risk-neutral potential bidders with the same discount rate r who have the expertise to exercise the option at a cost θ_i , $i = 1, 2, \dots, N$, which are i.i.d. with positive support $[\underline{\theta}, \bar{\theta}]$, where $0 \leq \underline{\theta} < \bar{\theta} \leq \infty$.⁵ Denote the cumulative distribution and density function by $F(\theta)$ and $f(\theta)$, respectively.

Similar to DeMarzo, Kremer, and Skrzypacz (2005) and Sogo, Bernhardt, and Liu (2016), a bidder has to pay an initial investment $K \geq 0$ upon winning the auction, which we can interpret as the initial resources the project requires, such as illiquid human capital, the legal cost of contracting, or simply his opportunity cost.⁶ All results hold qualitatively even when K is infinitesimal. K serves two roles in the model: (i) it allows bidder participation to be endogenous even absent reserve prices, which in turn allows me to discuss under general conditions how auction timing affects participation; (ii) it breaks the indifference in auction

⁵With symmetric bidders, unless necessary, I drop the i in θ_i when referring to a generic bidder's type.

⁶In practice, the seller may have a cost at the auction or the bidders may incur an entry cost. Their effects are similar to K , as I discuss in Section 5. For clarity, I refer to the seller as female and the bidders as male.

timing, and leads to unambiguous characterization of the welfare consequence of the seller's auction timing.⁷ The project operated by type θ_i then produces cash flows whose present value at the time of option exercise t is $X_t - \theta_i$.

When the auction is held at time t_a , bidders compete by offering cash bids. To make the comparison between this auction stage and the classical static auction settings (e.g., benchmark models in McAfee and McMillan, 1987) transparent, I denote the bidder- i 's valuation of the real option at the auction by W_i , which has one-to-one correspondence with θ_i and generally depends on the auction timing. I denote its cdf and pdf by $F_W(W)$ and $f_W(W)$. Regularity condition of the distribution of bidders' valuation W_i , i.e., $W_i - \frac{1-F_W(W_i)}{f_W(W_i)}$ is increasing, is standard in the auction literature (e.g., Myerson (1981)) to avoid invoking the complicated "ironing" arguments. To compare apples to apples when relating to well-known results in auction theory, I impose regularity condition on W_i regardless of the time of holding the auction.

Finally, welfare in this paper is defined by the total payoff to the seller and bidders, and efficiency in this paper means constrained efficiency from a social planner's perspective; that is, welfare maximizing under the same informational or institutional constraints as individual agents making decisions.

In sum, the agents interact in continuous time as shown in Figure 1.

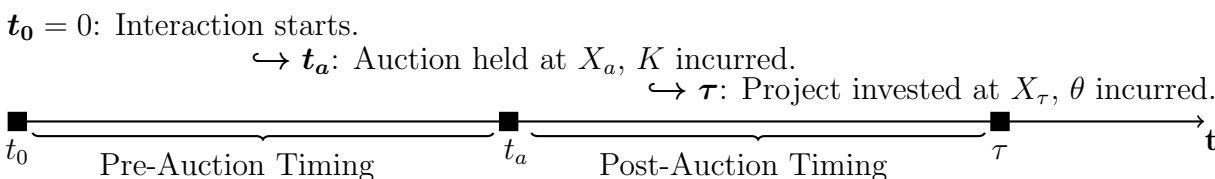


Figure 1: Timeline

I first consider formal auctions in which the seller has full commitment to auction design (both rules and timing), and discuss in Section 4 bidder initiation and negotiation absent such commitment. Within formal auctions, I focus on first-price auctions (FPAs) and second-price auctions (SPAs) in which the bidder with the highest bid wins and pays the highest bid or the second-highest bid, provided it exceeds the reserve price the seller endogenously

⁷Notice that $K > 0$ here also conveniently ensures that the worst type of bidder has zero valuation, consistent with the typical assumption in the auction literature that bidder types always have zero lower support. Absent K , no matter what θ is, a bidder always has strictly positive valuation of a real option.

sets.⁸ They also implement the optimal auction, as I discuss shortly.

3 Endogenous Auction Timing

To analyze the dynamics, I work backward to first solve for the optimal investment strategy for the winning bidder, derive the bidding equilibrium given the bidders' valuations based on their investment strategies, and then study the impact of the seller's strategic timing of the auction.

3.1 Optimal Stopping and Bidding Strategies

A bidder of type θ owns the project entirely upon winning, and optimally develops the project at time $t \geq t_a$ to maximize $\mathbb{E}[e^{-r(t-t_a)}(X_t - \theta)]$. Due to the irreversible nature of option exercise, the optimal strategy for this standard problem involves immediate investment upon reaching $X^*(\theta) \equiv \frac{\beta}{\beta-1}\theta$ (e.g. McDonald and Siegel, 1986; Dixit and Pindyck, 1994). Let X_a denote the cash-flow level when the auction is held. The value of the investment option W and $X^*(\theta)$ depends on t only through X_a :

$$X_a^*(\theta) = \max \{X_a, X^*(\theta)\}, \quad \text{where} \quad \beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1, \quad (2)$$

and $W(X_a; \theta) = D(X_a; X^*(\theta))[X_a^*(\theta) - \theta] - K, \quad \text{where} \quad D(X; X') = \left(\frac{\min\{X, X'\}}{X'}\right)^\beta.$

Note that $D(X_i; X')$ corresponds to the time- t price of an Arrow-Debreu security that pays one dollar at the first instance since time t that X reaches the region $[X', \infty)$ is reached. The option value of the project is simply the total value of Arrow-Debreu securities that replicate the payoff of the investment option at exercise.

Bidder i 's private valuation is then $W(X_a; \theta_i)$, which decreases in θ_i . For any given X_a , we can simply view the auction as a standard auction with private valuations $W_i = W(X_a; \theta_i)$, and agents' bidding strategies are the same as those in standard cash FPAs and SPAs. In particular, in SPAs a bidder of type θ bids $W(X_a; \theta)$ under weakly undominated strategy.

⁸FPAs and SPAs are the most prevalent form of auctions. Other common forms such as Dutch auctions or English auctions are equivalent to FPAs and SPAs respectively. When it comes to bidder participation, I discuss both the case without reserve price and the case in which the seller endogenously sets a reserve price.

From classic theories of auction (Vickrey, 1961; Myerson, 1981; Krishna, 2009), at the time of auction, we know FPAs and SPAs generate equivalent revenues to the seller, and allocate the project to type $\theta_{(1)}$ if $\theta_{(1)} \leq \hat{\theta}$, where $\theta_{(j)}$ is the j th lowest realized θ , and $\hat{\theta}$ is the cutoff-type that participates. In general, $\hat{\theta}$ depends on X_a and is endogenized by bidders' individual rationality (IR) constraint of participation.

Note that the regularity condition on W_i s limits the set of distributions on θ , but is not very restrictive. One sufficient condition for regularity for all X_a is $\frac{\theta}{\beta-1} - \frac{F(\theta)}{f(\theta)}$ being decreasing, which I assume for the remainder of the paper for simplicity. This is satisfied, for example, by all uniform distributions when $\beta \geq 2$. I also assume that $X_0 < X^*(\underline{\theta})$ so that auction timing and option exercise decisions are non-trivial.

3.2 Seller's Real Option and Control

Given the equilibrium bids and option exercise, a seller chooses auction timing t_a to maximize her expected revenue. To analyze the seller's optimal auction timing, we need to first examine her expected revenue from the auction. To this end, I use the Direct Revelation Principle. Let $Q(\tilde{W}_i, W_{-i}) = Q(\tilde{\theta}_i, \theta_{-i})$ be the probability of allocating the project to bidder i , when bidder i reports θ_i and others report types θ_{-i} (a vector). Denote $\hat{W} = W(X_a; \hat{\theta})$, $W_i = W(X_a; \theta_i)$, and $W = W(X_a; \theta)$. Then from Myerson (1981) and Krishna (2009), in any incentive compatible and individually rational mechanism the expected bidder- i payoff, $U(\theta_i)$, must equal

$$\mathbb{E}_{W_{-i}} \left[\int_{\hat{W}}^{W_i} Q(W, W_{-i}) dW \right] + U(\hat{\theta}) = \mathbb{E}_{\theta_{-i}} \left[\int_{\theta_i}^{\hat{\theta}} Q(\theta, \theta_{-i}) D(X_a; X^*(\theta)) d\theta \right] + U(\hat{\theta}), \quad (3)$$

and satisfies $U(\hat{\theta}) \geq 0$, where $W = W(X_a; \theta)$ is the valuation of an agent with cost θ and $\hat{W} = W(X_a; \hat{\theta})$ is the cut-off type's valuation. This leads to the following lemma.

Lemma 1. *The seller's $t = 0$ expected revenue in either a FPA or SPA timed at t_a is given by*

$$\mathbb{E} \left[e^{-rt_a} \mathbb{1}_{\{\theta_{(1)} \leq \hat{\theta}\}} \left(D(X_a; X^*(\theta_{(1)})) [X_a^*(\theta_{(1)}) - z(\theta_{(1)})] - K \right) \right], \quad (4)$$

where $\theta_{(1)}$ is the smallest realized cost.

The term e^{-rt_a} simply discounts the auction revenue to its present value; $\mathbb{1}_{\{\theta_{(1)} \leq \hat{\theta}\}}$ indicates

that the best type can afford the reserve price and participates. The surplus upon option exercise is $X_a^*(\theta_{(1)}) - \theta_{(1)}$, but the seller's payoff depends on the "virtual valuation" of the best type, $X_a^*(\theta_{(1)}) - z(\theta_{(1)})$, rather than the actual valuation (Bulow and Roberts, 1989).

To interpret the virtual valuation, note from (3) that the seller pays bidder i at t_a an expected informational rent of $\mathbb{E}_{W_{-i}} \left[\int_{\hat{W}}^{W_i} Q(W, W_{-i}) dW \right]$, which is equivalent to an added cost option exercise cost $F(\theta_{(1)})/f(\theta_{(1)})$. In other words, the seller essentially owns the best type's real option with a stochastic "virtual strike price" $z(\theta_{(1)})$. In general, the winning bidder's optimal investment timing differs from the seller's by an amount dependent on $F(\theta_{(1)})/f(\theta_{(1)})$.

If the seller were to dictate the option exercise, she would prefer an optimal exercise at $X^*(z(\theta_{(1)}))$ which is weakly bigger than the winning bidder's actual exercise threshold $X^*(\theta_{(1)})$. Yet the winning bidder has ownership and thus control over the option exercise after the auction. Anticipating this, the seller may want to delay holding the auction at an $X_i > X^*(\theta_{(1)})$ for some possible values of $\theta_{(1)}$. Because the winning bidder cannot go back in time to exercise the option at the lower threshold $X^*(\theta_{(1)})$, this irreversibility gives the seller partial control over option exercise — a key insight of the paper I repeatedly come back to.

Another force at play is that a winning bidder's initial investment K has a one-to-one map to a seller cost of holding the auction at X_a because the bidder would shade down the bid due to its reduced project valuation. The seller therefore also times the auction to delay the incidence of K . In sum, the seller may want to delay an auction to postpone incurring K and to control the winning bidder's option exercise.

3.3 Endogenous Participation and Reserve Price

Without reserve price, increasing X_a obviously relaxes bidders' individual rationality of participation. In particular, when $\hat{\theta} < \bar{\theta}$, the marginal type that participates satisfies

$$D(X_a; X^*(\hat{\theta}))[X_a^*(\hat{\theta}) - \hat{\theta}] - K = 0, \quad (5)$$

Obviously, $\hat{\theta}$ strictly increases in X_a .

When the seller sets a reserve price, it follows directly (e.g., from Myerson, 1981; Krishna, 2009) that her optimal reserve price under FPAs and SPAs are the same, and the cutoff type

contributes zero revenue upon winning. The cases with an entry fee are similar.

Lemma 2. *The seller optimally sets a reserve price $R(X_a) = W(X_a; \hat{\theta})$ in FPAs and SPAs, where $\hat{\theta} = \max\{\theta \in [\underline{\theta}, \bar{\theta}] : D(X_a; X^*(\theta))[X_a^*(\theta) - z(\theta)] - K \geq 0\}$ is unique.*

The condition on the marginal bidder is more stringent than (5) because $z(\theta) > \theta$ in general, and thus more bidders are excluded. Auction timing again has a strong impact on bidder participation.

Proposition 1. *Before reaching, $\bar{\theta}$, $\hat{\theta}$ strictly increases in X_a .*

Delaying auctions to a higher cash flow level relaxes marginal participants' constraints and on average results in more bidders placing bids. Moreover, delaying auctions strictly increases the optimal reserve price.

Corollary 1. *The optimal reserve price R always increases with X_a .*

It might appear contradictory that a higher X_a leads to both higher R and $\hat{\theta}$, because higher reserve price typically excludes more bidders. However, a higher X_a also means the bidders value the project more at the time of the auction, which allows more participation even though they face a higher reserve price.

3.4 Optimal Timing of Auctions

Thus far, I have shown how auction timing can help the seller gain partial control over the option exercise, delaying the incidence of initial investment K , and alter bidders' endogenous participation. The three effects interact to give the key result regarding auction timing.

Proposition 2. *An optimal threshold strategy for timing an auction with reserve price exists. The seller inefficiently delays the auction for any $K > 0$.*

From the seller's perspective, she owns a real option whose exercise she partially controls through auction timing. Option values only erode for both the seller and the bidders when X_a increases beyond $X^*(z(\bar{\theta}) + K) = \frac{\beta}{\beta-1}(z(\bar{\theta}) + K)$; the seller also strictly prefers to delay auction when $X_t \leq X^*(\underline{\theta} + K) = \frac{\beta}{\beta-1}(\underline{\theta} + K)$ because almost surely it discounts the cost K without hurting the seller's option value regardless of what the virtual strike price

$z(\theta_{(1)})$ takes on. The seller therefore holds the auction at $X_a \in (X^*(\underline{\theta} + K), X^*(z(\bar{\theta}) + K))$, which also implies that she never sells the project when she expects no chance of immediate investment.

We can alternatively interpret the result by looking at the distribution of bidders' time-zero valuation of the project, $D(X_0; X_a)W(X_a; \theta_i)$. Even though the distribution of θ_i is invariant, that of time-zero valuation depends on X_a . By delaying the auction slightly when $X_a > X^*(\underline{\theta})$, the seller effectively changes the time-zero valuation $D(X_0; X_a)W(X_a; \theta_i)$ over the interval $\theta \in [\underline{\theta}, \frac{\beta-1}{\beta}X_a]$ from $D(X_0; X^*(\theta_i)) [X^*(\theta_i) - \theta_i] - D(X_0; X_a)K$ to $D(X_0; X_a)[X_a - \theta_i] - D(X_0; X_a)K$. Then we have

$$\frac{\partial^2}{\partial X_a \partial \theta} [D(X_0; X_a)W(X_a; \theta)] = \begin{cases} 0, & \text{if } X_a \leq \frac{\beta}{\beta-1}\theta \\ \beta X_0^\beta X_a^{-\beta-1} > 0 & \text{otherwise} \end{cases} \quad (6)$$

which implies that as X_a increases, the difference in time-zero valuations for any two θ' and θ'' remains constant if both θ' and θ'' exceed $\frac{\beta-1}{\beta}X_a$ (both real options' values are eroded to the same extent), and strictly decreases otherwise (the more valuable real option's value is eroded). In other words, the distribution of $D(X_0; X_a)W(X_a; \theta_i)$ over $\theta \in [\underline{\theta}, \frac{\beta-1}{\beta}X_a]$ becomes compressed.

Over this interval, the time-zero option value to a seller with a higher X_a is reduced because it would be exercised beyond what is socially optimal, but the more compressed valuation distribution encourages bidder competition and reduces the information rent the seller has to pay to ensure incentive compatibility in the auction mechanism (types more easily separate). This can be seen from the time-zero expected bidder payoff, $D(X_0; X_a)U(\theta_i)$ (see (3)): $D(X_0; X_a)dW = D(X_0; X_a)D(X_a; X^*(\theta))d\theta$ becomes smaller when $X_a > X^*(\theta)$. Consequently, the time-zero expected buyer payoff $D(X_0; X_a)U(\theta_i)$ is also smaller, allowing the seller to extract more rent. A higher X_a also expands the support of the distribution and encourages bidder participation.

Figure 2 illustrates these effects. It shows the distribution of the time-zero valuations, $D(X_0; X_a)W(X_a; \theta)$, where θ follows $f(\theta)$ on $[\underline{\theta}, \bar{\theta}]$. The shaded region corresponds to the valuation distribution of participating bidders, though I did show the density of valuations corresponding to all $\theta \leq \bar{\theta}$. The three panels use $X_a = 18, 20, 22$ respectively. As we increase X_a in the next two panels, the distribution shifts to the right, due to the discounting of

K. The density closer to $D(X_0; X_a)W(X_a; \underline{\theta})$ also changes due to some option values being compressed. Finally, the cutoff valuation in participation, $D(X_0; X_a)W(X_a; \hat{\theta})$, changes too because higher X_a leads to greater participation and higher reserve price.

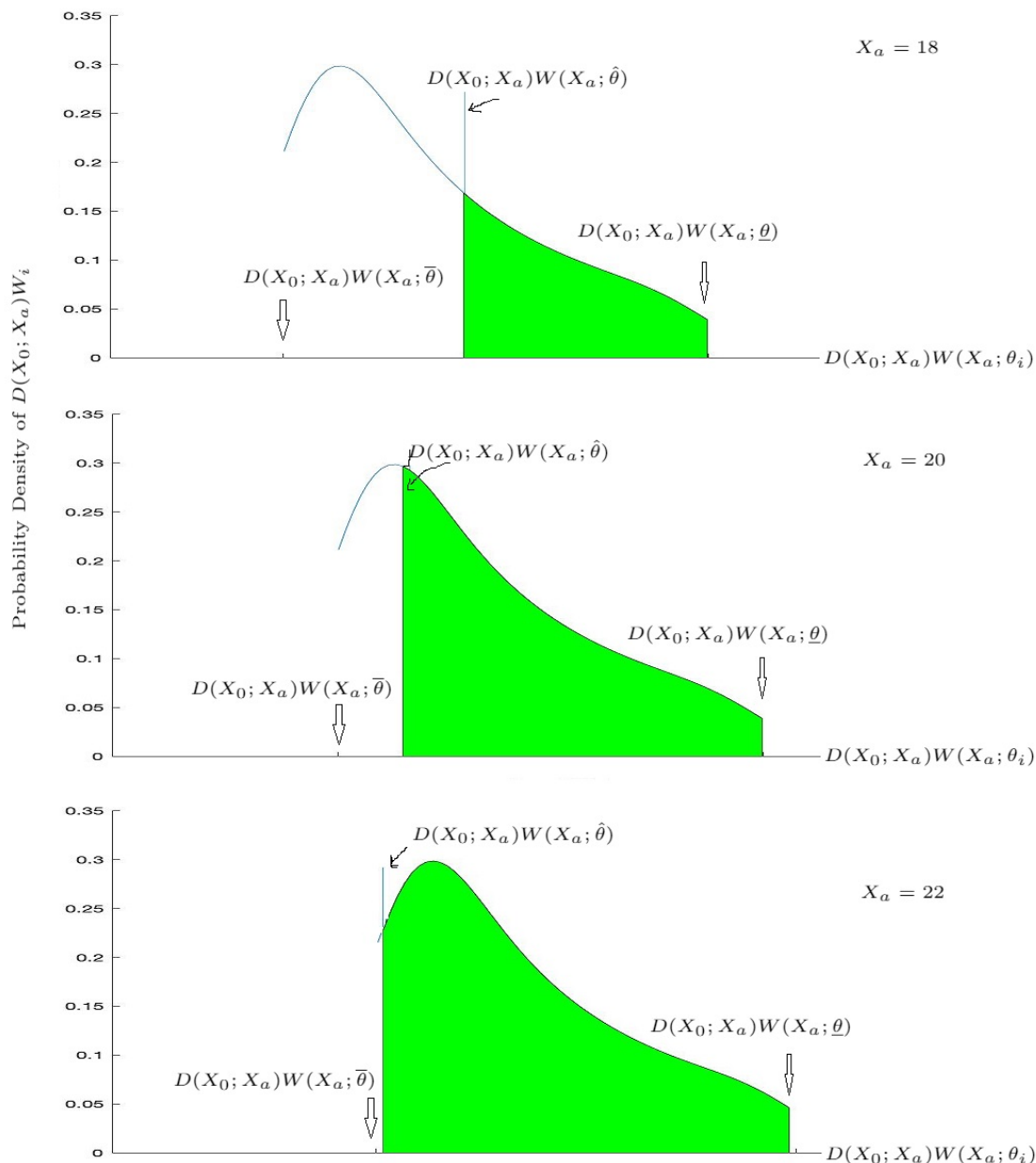


Figure 2: PDF of Time-zero Valuations.

$X_0 = 18$, $\theta_i \sim Normal(15, 3)$, truncated to $[\underline{\theta} = 10, \bar{\theta} = 30]$, $r = 0.06$, $\mu = 0.01$, $\sigma = 0.2$, $K = 3.5$.
The three panels correspond to $X_a = 18$, $X_a = 20$, and $X_a = 22$ respectively.

When X_a is low, the marginal benefit from additional rent extraction, bidder participa-

tion, and delaying K dominates the erosion of the time-zero option value for some bidders; but when X_a is large, the option value erosion dominates. Therefore, the seller chooses X_a to balance these forces without knowing the realization of bidder types. The general tradeoff holds even when K is infinitesimally small.⁹

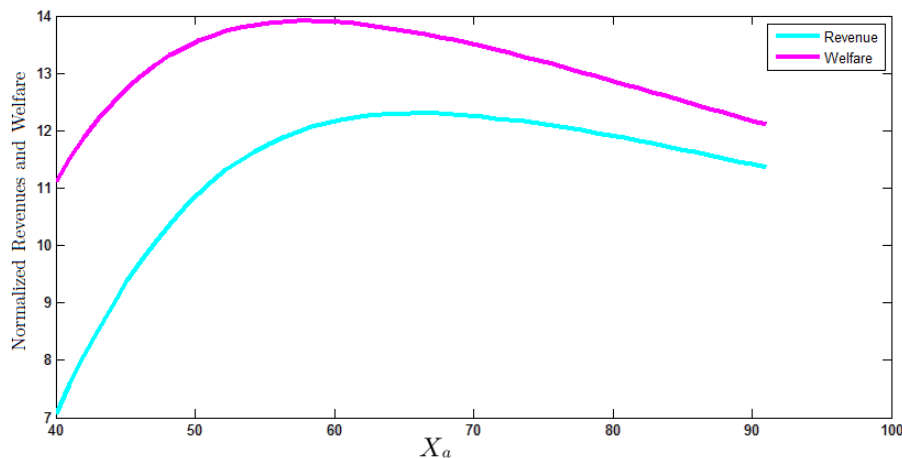


Figure 3: Revenues and Welfare for cash auctions following threshold timing X_a . 200,000 simulations for $\theta \sim Unif[10, 40]$, $r = 0.06$, $\mu = 0.01$, $\sigma = 0.2$, $Y = 15$, $N = 7$, $X_0 = 40$. Welfare-maximizing auction timing threshold is lower than revenue-maximizing timing.

The fact that the seller’s virtual valuation is lower than a social planner’s leads to inefficient auction delays. Figure 3 illustrates this effect by plotting time zero present values of the expected revenues and welfare from cash auctions held when X_t first reaches X_a .

A social planner with the same information as the seller would hold the auction at an $X_t \in (X^*(\underline{\theta} + K), X^*(\bar{\theta} + K))$, balancing the cost of inefficient option exercise with the benefit of delaying the incidence of K . Because the seller chooses a bigger X_a , the real option exercise is inefficiently late with greater probability (for more types of θ).

This is a striking result because FPAs and SPAs are believed to be efficient (at least the ones without reserve prices). Even though reserve prices could distort allocative efficiency, cash bids do not distort post-auction incentives. Yet inefficient auction delays can lead to more inefficient option exercises. Given that in practice most auctions involve objects or

⁹However, when $K = 0$ exactly, the social optimal auction timing is indeterminate and can be any $X_a \leq X^*(\underline{\theta})$. The seller’s auction timing may be earlier or later than a social planner’s. In fact, without reserve prices, one can tell the seller’s timing is also efficient. The intuition is that without expanding participation that contributes weakly positively to the seller’s revenue, compressing the time-zero distribution of valuations from the top may only reduce the seller’s revenue.

projects with optionality in them, the importance of auction timing cannot be relegated into the backseat in auction theory.

3.5 Optimal Design and Multi-dimensional Inefficiency

We know from the literature that either FPAs or SPAs with optimal reserve price implements the optimal auction mechanism (Myerson, 1981; Krishna, 2009). Therefore,

Corollary 2. *The seller's optimal selling mechanism through cash auction can be implemented by a FPA or SPA with an optimal reserve price $R(X_a)$ described in Lemma 2, optimally timed as in Proposition 2. It entails inefficient option allocation and exercise.*

I use the qualifying words “cash” because there exists better mechanisms if we allow the agents to bid contingent securities, which I discuss in Section 5 and the appendix.

The optimal auction here involves setting a reserve price and auction timing. These results relate to Myerson (1981)'s analysis in a static setting regarding the wedge between the seller's revenue and welfare: Using a reserve price to exclude bidders can boost competition to increase the seller's revenue, but inefficiently distorts the allocation of the project.

Auction timing leads to several novel features in an optimal auction. Although the seller still excludes bidders, the auction is held under better market conditions (higher X_a), which encourages participation and mitigates the exclusion. This implies that in real life one may not see sellers excluding bidders as much using entry fees or reserve prices, because they have the alternative tool of choosing a more propitious time to hold the auction. For example, an entrepreneur selling a start-up seldom excludes potential acquirers, but rather waits for the product to have higher valuations before going onto the market.

While delaying auction can reduce the allocative inefficiency, it causes option exercises beyond $X^*(\theta_{(1)})$ — the winning bidder's preferred exercise timing. The delay is partially efficient because it delays the initial investment K the bidder has to pay. This means K alone does not lead to greater inefficiency because even a social planner would time the auction beyond $X^*(\theta)$ for some range of θ . It is the additional delay due to the seller's desire to reduce information rent and increase bidder participation that leads to socially inefficient delays in option exercises at $X_t > X^*(\theta_{(1)} + K)$. Even as $K \rightarrow 0$, so distortion remains.

Evidently, the wedge between the seller's revenue and welfare with endogenous auction

timing is multi-dimensional. For regulators concerned with welfare, auction timing is an integral consideration. This is further complicated when bidders can initiate an auction, as I discuss next.

4 Informal Auctions and Bidder Initiation

Many economic interactions such as corporate takeovers, competition for supply contracts, and talent recruitment have characteristics of auctions because buyers are competing with one another to make offers. Yet the seller may not have full control over the selling mechanism due to the lack of commitment that McAfee and McMillan (1987) discusses. In particular, a bidder may approach the seller with an offer any time, and a seller can negotiate, wait, and solicit further offers. For example, in M&As and patent sales, bidders often can initiate a sale process resembling an auction, as described in Fidrmuc, Roosenboom, Paap, and Teunissen (2012) and Gorbenko and Malenko (2017).¹⁰ I call them informal auctions.

As McAdams and Schwarz (2007) point out, in practice committing to sealed bids in informal auctions is difficult, especially in corporate acquisitions.¹¹ The board of directors of a target firm has to disclose all bids to shareholders, and considers subsequent offers to avoid shareholder lawsuits. Therefore informal auctions in real life either entail multiple rounds of negotiations and repeated communications, or manifest themselves in two-stage auctions routinely used in privatization, takeover, and merger and acquisitions.¹² The former resembles an informal English auction in which buyers adjust their bids until one winner emerges; Perry, Wolfstetter, and Zamir (2000) show the latter are typically robust mechanisms equivalent to an English auction.

I therefore treat informal auction initiation as a dynamic game in which a bidder or the

¹⁰When an asset of a Delaware corporation is for sale, the Revlon rule imposes upon directors a duty to solicit competitive bids to maximize shareholders' value. Although many takeovers occur after one-on-one negotiations, even in such cases potential rival buyers present latent competition (Aktas, De Bodt, and Roll, 2010), and in recent years "Go-shop negotiations"—an auction follows a negotiation in the initial round, as with, for example, the sale of CKE Restaurants in 2009—became more frequent.

¹¹In "Lawsuit Seeks to Block Sale of G.M. Building", New York Times, September 20, 2003, Charles Bagli documents how General Motors entertained a late offer after auctioning its Manhattan building in a first-price auction. Even in formal auctions, such a commitment is often hard to maintain.

¹²In the first stage, agents simultaneously submit sealed bids. Lower bids that fail to pass the first-stage are publicly revealed, while some highest bidders continue to a second sealed-bid auction where each bidder is bounded below by the first-stage bid. See, for example, Perry, Wolfstetter, and Zamir (2000) and Frankel (2011).

seller at any $t > 0$ before the project is sold can trigger an English auction immediately to allocate the project and ends the initiation game. Because it is an undominated strategy to bid one's true valuation in an English auction, an initiating bidder must offer at least the highest bid the seller solicits from other bidders. The equilibrium concept is Perfect Bayesian and as is standard in the literature, I focus on symmetric equilibria with a weakly monotone threshold for initiation; that is, if $\theta_i < \theta_j$, the initiation threshold for bidder i is weakly lower than that for bidder j .

In such an equilibrium, all agents dynamically update their beliefs about the distribution of types present in the absence of any initiation. Once a bidder initiates at X_a , the distribution of bidder valuations is again updated to have an atomic mass at the type that would initiate at X_a in equilibrium, i.e., $X_I^{-1}(X_a)$, for the initiating bidder.

A bidding equilibrium with FPA involves asymmetric bidders if the auction is bidder-initiated and does not survive bidders' offer adjustments and renegotiation in practice.¹³ Asymmetries among bidders do not affect bidding behavior in SPAs, however; it is still a weakly dominant strategy for each bidder to bid his or her value (Krishna (2009)). Therefore, focusing on SPAs and English auctions also helps us illuminate the key economic mechanism under relatively transparent bidding and option exercise strategies.

More formally, an initiation equilibrium is a collection of thresholds $\{X_S, \{X_I(\theta), \theta \in [\underline{\theta}, \bar{\theta}]\}\}$ upon hitting which the agent initiates the auction if it has not been initiated yet. The threshold $X_I(\theta)$ for a bidder of type θ satisfies,

$$X_I(\theta) = \operatorname{argmax}_{X_a \geq X_0} \mathbb{E} \left[D(X_0; \tilde{X}) \left[W(\tilde{X}; \theta) - [W(\tilde{X}; \theta_{(1)}^{N-1})]^+ \right]^+ \middle| \tilde{X} \leq X_a \right] + \mathbb{E} \left[D(X_0; X_a) \left[W(X_a; \theta) - [W(X_a; \theta_{(1)}^{N-1})]^+ \right]^+ \middle| \tilde{X} > X_a \right], \quad (7)$$

where $\tilde{X} = \min \left\{ X_I \left(\theta_{(1)}^{N-1} \right), X_S \right\}$ is the cash flow threshold at which the auction is first initiated, and $\theta_{(1)}^{N-1}$ is the first-order statistics of the remaining $N-1$ bidders. Note that $\theta_{(1)}^{N-1}$ has the lowest initiation thresholds among all other bidders in an equilibrium with monotone

¹³In FPAs, bidding strategies depend on the dynamically updated beliefs about the types present. In a monotone equilibrium, a better type that initiates reveals his type, and is not guaranteed to win unless he bids his own valuation, because a slightly worse type can outbid him. But once he wins, he would want to lower the payment to the next highest bid, essentially transforming the sale into a second-price auction. For a detailed discussion of bidder initiation in FPAs absent post-auction option exercises, see Gorbenko and Malenko (2016).

initiation. The first term in (7) represents the expected payoff when another bidder or the seller initiates before he does, while the second term represents the expected payoff when this bidder initiates at X_a before others.

Similarly X_S has to maximize the seller's expected payoff, that is,

$$X_S = \operatorname{argmax}_{X_a \geq X_0} \mathbb{E} \left[D(X_0; X_I(\theta_{(1)})) [W(X_I(\theta_{(1)}); \theta_{(2)})]^+ \mid X_I(\theta_{(1)}) \leq X_a \right] + \mathbb{E} \left[D(X_0; X_a) [W(X_a; \theta_{(2)})]^+ \mid X_I(\theta_{(1)}) > X_a \right], \quad (8)$$

where the first term represents the expected payoff when a bidder initiates before he does, while the second term represents the expected payoff when the seller initiates at X_a before others.

Is there still inefficient delay in auctions when bidders can initiate? How does it alter the seller's revenue and welfare? These are not only of theoretical interests, but also have empirical relevance that I discuss shortly after I characterize endogenous auction initiation.

4.1 Endogenous Initiation by Sellers or Bidders

From (7) and (8), we know that the seller times the auction to maximize the second-highest valuation, whereas a bidder times the auction to maximize the present value of informational rent (difference between his valuation and the second-highest valuation).¹⁴ The latter starts to erode earlier than the former as the initiation threshold X_a increases. Therefore, the seller always waits in such an equilibrium, as the next proposition describes.

Proposition 3. *An informal auction admits an essentially unique auction timing equilibrium. The seller never initiates and a bidder of type θ initiates with threshold $\max\{X_I(\theta), X_0\}$, where $X_I(\theta)$ is weakly increasing in θ and uniquely solves*

$$\int_{\theta}^{\hat{\theta}} \frac{d}{dX} \frac{W(X; \theta) - W(X; \theta')}{X^{\beta}} \Big|_{X=X_I} f(\theta') [1 - F(\theta')]^{N-2} d\theta' = 0, \quad (9)$$

and $\hat{\theta}$ solves $W(X_I; \hat{\theta}) = 0$.

¹⁴Had we allowed the seller to still commit to a reserve price in a bidder-initiated auction, the seller infers the best type present and fully squeezes the bidder's rent, which reduces our analysis trivial. The case when the seller sets a fixed reserve price would not change the main results either, but is left out for brevity.

If by cash flow level X the auction has not been initiated, everyone updates their beliefs about types that are present. The seller times the auction to maximize the second-highest valuation, conditional on the support of bidder types being $(X_I^{-1}(X), \bar{\theta}]$. The bidder of the best type now has $\theta = X_I^{-1}(X)$. He times the auction to maximize the present value of his informational rent, therefore he would initiate immediately upon hitting $X_t > X^*(X_I^{-1}(X))$. However, Proposition 2 would prescribe for the seller to initiate at an $X_a > X^*(X_I^{-1}(X))$, which is a weakly higher threshold than that of the bidder of the best type at the moment.

A bidder gets the difference between his valuation and the second highest valuation, and does not bear the cost of K unless he is the only participant, because this investment cost impacts the highest valuation and the second highest valuation in the same way: a higher K increases the winner's cost, but if the second highest bidder's valuation is positive, a higher K lowers that value and thus the payment the winner makes. As such, the initiation threshold $X_I(\theta)$ is lower in general than that of a social planner knowing this type's presence $X^*(\theta + K)$. In other words,

Corollary 3. *Bidder-initiated informal auctions are weakly accelerated relative to that under an initiation achieving the first-best social welfare.*

Although bidder initiation accelerate auction timing, the bidder's option value starts to be eroded only after $X_t \geq X^*(\theta)$, so $X_I(\theta) \geq X^*(\theta)$. This implies that once the auction is initiated, the bidder wins and exercises the option right away. In contrast, in auctions excluding bidder initiation, a seller holds the auction only when the second best type's option starts to erode *in expectation*. Hence it is possible that the "realized least" exercise cost, $\theta_{(1)}$ is large enough that the option would not be exercised right away.

Because bidder would not initiate until the investment threshold for his real option is reached, the next corollary links auction initiation to option exercise, and is evidence that endogenous auction initiation and timing have material impact on equilibrium real outcomes:

Corollary 4. *The real option is exercised more quickly after the auction allowing bidder initiation than after those only allowing seller's auction timing.*

Here quickness refers to how soon after the auction the real option is exercised. The corollary simply follows from the fact that in a seller-initiated auction, it is possible that the winner has a high θ and waits to exercise the real option, whereas the waiting time

in auctions allowing bidder initiation is zero because bidders exercise the option immediately after initiation. Also note that a bidder knows his θ and would time the auction only when he is ready to exercise the option, for otherwise he can delay the incidence of initial investment K without missing his opportune option exercise. Bidder initiation therefore provides a channel for bidders' private information to affect auction timing. Next I show that incorporating bidders' private information can be useful for the social welfare.

Corollary 5. *As $K \rightarrow 0$, bidder initiation and option exercise approach the first-best in terms of social efficiency, and is more efficient than an auction forbidding bidder initiation.*

Note that the second half of the statement is equivalent to saying that for auctions allowing seller initiation only, $K \rightarrow 0$ does not generate as much social surplus. This wedge exactly comes from the fact that bidder initiation incorporates private information of θ_i , $i = 1, 2, \dots, N$ which makes auction timing by bidder i more efficient. Full efficiency is reached as $K \rightarrow 0$. In contrast, in auctions forbidding bidder initiation, the seller times the auction without information about the best realized bidder type, and the virtual strike price makes her always want to weakly delay the auction compared to a social planner.

Admittedly, to relate these theoretical findings to a specific application such as M&As requires enriching the model with more realistic features.¹⁵ Nevertheless, they provide insights future studies could build on and are broadly consistent with empirical observations. For example, patent holders rarely organize an auction and instead are often approached by acquirers. Also, (Fidrmuc, Roosenboom, Paap, and Teunissen, 2012) document almost 80% of M&As are bidder-initiated. To the extent that they are mostly informal auctions, they corroborate the model prediction that sellers never initiate when bidders can initiate.¹⁶

¹⁵For example, Wang (2018) provides a more careful study on go-shop negotiations, modeling all stages involved and allowing correlated valuations that is more accurate for modeling financial bidders. Note my baseline model is more applicable to strategic acquisitions where bidders are more likely to have private information regarding valuation than in financial acquisitions.

¹⁶As another example, Aktas, De Bodt, and Roll (2010) and Masulis and Simsir (2015) document that bidding premium in M&As crucially depends on whether a bidder or a target initiates; target firms receive lower bid premium in target-initiated deals after controlling for observables. This can be reconciled in the current setting without resorting to two-sided private information as in Chen and Wang (2015): if one views bidding premium as the amount the highest valuation exceeds the average, then it is higher in auctions with bidder initiation because the distribution of valuations has not been compressed from the top, given that the best type under bidder initiation uses an initiation threshold $X_a \leq X^*(\theta_{(1)} + K)$ which on average is lower than that in an auction timed by the seller only.

4.2 Initiation under Seller Approval

So far I allow bidders to initiate the auction immediately, which is a feature in prior studies such as Gorbenko and Malenko (2017, 2016). That said, what if a buyer cannot force the seller to hold an auction, especially when in practice the seller may incur a cost allocating the project, such as the loss of private benefit from control? I now relax the assumption and allow the seller to reject the initiating bidder's request to immediately auction the project. Instead, she can continue soliciting offers and allow offer adjustments, and allocate the project at a more opportune time. In other words, the seller has to approve a bidder's initiation for the project to be auctioned and allocated.

Because bidder types are time-invariant, the rank of bidder valuations does not change even if the seller waits longer. Therefore, no matter who initiates, the seller's time-zero expected payoff holding the auction at X_a is $\mathbb{E} \left[D(X_0; X_a) [W(X_a; \theta_{(2)})]^+ \right]$. If the project has not been sold yet and a bidder of type θ_i initiates at X' , one way to maximize her payoff is to hold the auction at $X_S(X') \equiv \max\{X', X^*(\theta_{(2)} + K)\}$ because the threshold $X^*(\theta_{(2)} + K)$ maximizes the second-best type's valuation from getting the real option and exercising it. Notice that this bidder's initiation could depend on his time- t information filtration at initiation.

If $X' \neq X^*(\theta_i + K)$, I argue that it is in bidder i 's best interest to initiate instead at $X^*(\theta_i + K)$ and make an offer attractive enough that the seller would hold the auction right away, regardless of other agents' initiation strategies. To see this, notice that his maximum payoff normalized to time-zero value is

$$\max_{X'} \mathbb{E} \left[D(X_0; X_S(X')) [W(X_S(X'); \theta_i) - [W(X_S(X'); \theta_{(2)})]^+]^+ \right]. \quad (10)$$

Basically he is no longer competing with the highest valuation from other bidders *at the moment of* initiation, but the highest valuation *at or after* initiation. Now on all the realizations that he is going to win (i.e., $\theta_i = \theta_{(1)}$), suppose $X' < X^*(\theta_i + K)$, then he benefits from waiting to reach $X^*(\theta_i + K)$ because it increases his option value without increasing the expected discounted amount he pays to the seller; suppose $X' > X^*(\theta_{(2)} + K)$, then his payoff would be $D(X_0; X') [\theta_{(2)} - \theta_i]$ and he would get a higher payoff had he chosen $X' = X^*(\theta_{(2)} + K)$; finally, suppose $X' \in [X^*(\theta_i + K), X^*(\theta_{(2)} + K)]$, his payoff would have been higher if

he initiates at $X^*(\theta_i + K)$ and offers to pay $D(X^*(\theta_i + K); X^*(\theta_{(2)} + K)) [W(X^*(\theta_{(2)} + K); \theta_{(2)})]^+$, because the seller would have no incentive to wait further and bidder i would get his highest option value.¹⁷

Therefore, in equilibrium, bidder i uses an initiation threshold $X^*(\theta_i + K)$ and makes an offer at the initiation to pay the seller a minimum of

$$D(X^*(\theta_i + K); X^*(\theta_{(2)} + K)) [W(X^*(\theta_{(2)} + K); \theta_{(2)})]^+ + \epsilon, \quad (11)$$

where ϵ is infinitesimal. The seller then has incentives to approve bidder i 's initiation immediately. Given this, the seller would not initiate in equilibrium because following the earlier argument, upon seeing no initiation up till X_t , she knows the best type present has a cost lower than $X_I^{-1}(X_t)$. Proposition 2 again implies that she would delay the auction past $X^*(X_I^{-1}(X_t) + K) = X_t$. The following proposition ensues.

Proposition 4. *When bidder initiation has to be approved by the seller, bidders always initiate in equilibrium, and the seller always approves. A bidder of type θ uses an initiation threshold $X^*(\theta + K)$, and exercises the option immediately upon winning.*

For any given $\theta_{(1)}$, the first-best initiation threshold in terms of social welfare, X_a , maximizes $D(X_0; X_a) [W(X_a; \theta_{(1)}) - K]$. This means the auction initiation threshold is $X^*(\theta_{(1)} + K)$, which coincides with the bidder initiation thresholds in an equilibrium with monotone initiation.

Corollary 6. *When bidder initiation has to be approved by the seller, auction timing and option exercise achieve full social efficiency.*

Informal auctions therefore achieve the first-best in terms of social welfare, no matter what K is. Again, this result comes from the fact that bidder initiation incorporates private information of θ_i which makes auction timing by bidder i more efficient. In contrast to Corollary 5, full efficiency is reached even with $K > 0$ here. To see this, note that the

¹⁷One small caveat is that bidder i knows at his initiation $X^*(\theta_i + K)$ the realized $\theta_{(2)}$ at $X^*(\theta_{(2)} + K)$. He can simply look at the price at which all other bidders drop out. In addition, to encourage $\theta_{(2)}$ to participate even when $X^*(\theta_i + K) < \theta_{(2)} + K$, the initiating bidder has to offer to pay the initial investment if the winning bidder of type θ_j has $X^*(\theta_i + K) < \theta_j + K$. These are all realistic offers the initiating bidder could make.

initiation affects the incidence of K , but not the time-zero value of his payment to the seller. The initiating bidder thus takes K into consideration when timing the initiation.

Considering that an auction allowing bidder initiation is more efficient than one forbidding bidder initiation, and informal auctions allow more efficient timing through utilizing the bidders's information, a social planner should use informal auctions instead formal auctions, which has policy implications, for example, in oil lease auctions.¹⁸ Informal auctions or even formal auctions allowing bidder initiation also have the simplicity in implementation because they do not require the social planner to know the distribution of bidder types, and are thus compatible with the Wilson Doctrine (Wilson, 1985).

5 Discussions and Extensions

To clearly convey the intuition, I have abstracted away from some realistic features of auctions of real options. For example, bidders may have correlated real option exercise costs; the seller may need to incur a cost for holding the auction or have private benefits derived from the option exercise; bidders may bid in combinations of cash and contingent payments. To understand why the main economic insights apply to more general settings, I extend the model along all these dimensions.

5.1 Seller's Real Option under General Specification

Auctions with Interdependent Values. Let us consider an exercise cost function $\Theta(\theta_i, \theta_{-i})$ if bidder i exercises the real option, where the cost function is symmetric in other bidders' report types, and has positive derivative in θ_i denoted by Θ_1 that is uniformly bounded by a constant $A > 0$. In the baseline $\Theta(\theta_i, \theta_{-i}) = \theta_i$ and corresponds to private-value auctions; $\Theta(\theta_i, \theta_{-i}) = \sum_{j=1}^N \theta_j$ corresponds to a pure common value auction. This specification captures auctions with general inter-dependent payoffs. We require that $\Theta(\theta_i, \theta_{-i}) \geq \Theta(\theta_j, \theta_{-j})$ if $\theta_i > \theta_j$, so that $\theta_{(1)}$ still represents the best type. For tractability, I also require $\Theta(\theta_i, \theta_{-i})$ to be separable, that is $\Theta_1(\theta_i, \theta_{-i})$ is a function of θ_i only, which includes situations in classic

¹⁸The introduction of Area Wide Leasing in May 1983 for offshore drilling in the Gulf of Mexico eliminated nomination of tracts for auctions, which effectively changed the auction from allowing bidder initiation to forbidding bidder initiation. My model predicts that this change can cause inefficient auction timing and option exercises.

studies such as Bulow and Klemperer (2002).

Other costs of auction. The seller may incur a cost $Y \geq 0$ at the time of auction, which can be an overhead, effort cost of contacting potential bidders, pre-contract costs, or costs associated with revealing proprietary information to others (French and McCormick (1984); Hansen (2001); Gorbenko and Malenko (2017)), or her loss of private benefit of control, or even the option value of more efficient allocation of the asset when technology improves.¹⁹ The project is thus worth Y to her if it is never developed. This turns out to not affect the results because there is a one-to-one mapping between Y and K .

Seller's private payoff. The seller, namely the entrepreneur or an initial investor, may also have intrinsic preferences for the project's future that differ from the bidders. The inventor of a technology or a product might develop emotional attachment (Tjan (2011) and Matyszczyk (2015)), prefer an earlier commercialization, or favor later product shutdown; the manager of a firm may not fully internalize the reputation cost of the entrepreneur who originally founded the company, but instead care more about firm performances in her relatively short tenure.

In the same spirit as Grenadier, Malenko, and Malenko (2016) and Malenko and Tsoy (2017), I assume the seller derives a private non-pecuniary payoff of b when the option is exercised. As in these papers, b is common knowledge but non-contractible. Positive b indicates her preference for early option exercise; negative b indicates her preference for late option exercise. We note that b is really a measure of divergence in preference rather than a bias only for the seller. For example, the new manager of a firm may enjoy a private benefit from acquiring a target firm that would show up as a negative b in our model.

¹⁹Bleakley and Ferrie (2014) show that after an initial allocation of the frontier land in Georgia, land use took over a century to converge to post-allocation efficiency and land value was depressed by 20%. Another example is the FCC spectrum auction, where the government selling certain bandwidth to multiple firms has to consider the cost of losing the option to allocate it in the future to firms with better technology because even for the federal government, repurchasing the bandwidth is hard due to the well-known hold-up problem involved in multilateral bargaining. Y may be insignificant, especially when a winning bidder can contract with the seller to continue the original use before the option is exercised. But it could be large in many other cases. For example, the federal government typically auctions areas of land or sea involving multiple leases in a shared ecosystem, and cannot contract with individual winners to keep certain areas intact while allowing drilling in a neighboring tract. Due to political and ideological differences, national parks and environmental organizations are unlikely to collaborate with energy firms to maintain their operations before the energy firms start exploiting.

Bidding with securities. Finally, I allow the bidders to bid with contingent securities that pay the seller at the time of option exercise and are contingent on X_t at the real option exercise. Interested readers can read about the formal definitions of security bids and the concept of informal auctions with security bids in Appendix A.7. For now it is sufficient to know that security bids include all commonly used securities such as equity (bidding on shares), debt (bidding on face value), call option (bidding on strike price), and bonus with fixed royalty (bidding on upfront bonuses, as in oil tract auctions).

A general characterization. The following proposition is crucial in understanding why the intuition in the baseline model generalizes.

Proposition 5. *The seller’s auction timing strategy at t exists and involves the optimal stopping time $t_a \geq t$ that maximizes*

$$\mathbb{E}_t \left[e^{-rt_a} \mathbb{1}_{\{\theta_{(1)} \leq \hat{\theta}\}} \left[e^{-r(\tau_1^* - t_a)} \left(X_{\tau_1^*} - \theta_{(1)} - \frac{F(\theta_{(1)})}{f(\theta_{(1)})} \Theta_1(\theta_{(1)}) + b \right) - Y - K \right] \right], \quad (12)$$

where $\theta_{(1)}$ is the smallest realized cost, and τ_1^* is the bidder $\theta_{(1)}$ ’s optimal exercise of the option.

Note that $\tau_1^* = X_a^*(\theta_{(1)})$ in cash auctions, but could be different with security bids. Cong (2017) proves the existence of optimal option exercises with security payments.

From the proposition, it immediately follows that Y and K play similar roles in auction timing. More importantly, relative to the winning bidder, the seller faces a real option with an additional term in the “strike price,” $b - \frac{F(\theta_{(1)})}{f(\theta_{(1)})} \Theta_1(\theta_{(1)})$, which leads to a divergence in the preferred option exercise timing, just like in the baseline model.

5.2 Seller’s Private Payoff from Option Exercise

An emerging literature in dynamic corporate finance finds that the direction of the conflict of interest between sellers and buyers play an important role in timing decisions (Grenadier, Malenko, and Malenko, 2016; Guo, 2016). It is worth understanding how the divergence between the seller and bidders, b , affects auction timing and option exercise, especially whether its sign can lead to asymmetric outcomes.

Corollary 7. *As long as the seller does not have strong relative preference for early exercise, she inefficiently delays the auction and never sells the project when she expects no chance of immediate investment. Option exercises are almost surely inefficiently late when $b > 0$, but could be early or late when $b < 0$.*

Intuitively, a positive b reduces the “virtual strike price” the seller faces when a winning bidder exercises the option. The bigger b is, the less the seller delays the auction. But as long as $\frac{F(\theta_{(1)})}{f(\theta_{(1)})}\Theta_1(\theta_{(1)}) - b > 0$, the seller still prefers a later exercise time than the bidder. The irreversible nature of time still allows her to hold up the bidders, which forces the bidder to partially incorporate her preference on option exercise.

The winning bidder does not internalize the seller’s private payoff b when exercising the option, therefore the option exercise tends to be late when $b > 0$. When $b < 0$, the bidder tends to inefficiently accelerate the exercise, but the seller’s auction delay can mitigate this. Depending on which dominates, the exercise could be inefficiently early or late.

Corollary 8. *When the seller strongly prefers early exercise (b is sufficiently positive), the auction timing is socially efficient, but option exercises are still inefficiently late.*

The uni-directional nature of time flow again creates the asymmetry that timing the auction early would not help the seller control the option exercise because the winning bidder can always wait longer.

It should be clear that while the direction of the conflict of interest (timing here) still remains important as the literature shows, in the current setting it interacts with competitive bidding and bidder’s information rent, and its asymmetric impact is no longer based on a simple dichotomy of the direction of bias. The sign of b does not determine whether an auction is accelerated or delayed, nor whether the real option is exercised early or late. For example, in formal auctions, even when the seller prefers early sale and exercise of the option ($b > 0$), she may still delay the auction; even when a bidder prefers an earlier option exercise than the seller ($b < 0$), the option exercise could be late because of the seller’s delaying the auction. Informal auctions with sellers’ private payoff can be similarly analyzed and constitute interesting future work.

5.3 Timing Auctions with Security Bids

With security bids, “formal auctions” mean that the seller can commit to the security design and auction timing. Cong (2017) characterizes the bidding strategies in FPAs and SPAs for a fixed auction time t_a . In the appendix, we show that Lemma 2 and Proposition 1 still hold, and the proofs are only modified with a new stopping time by the winning bidder.²⁰

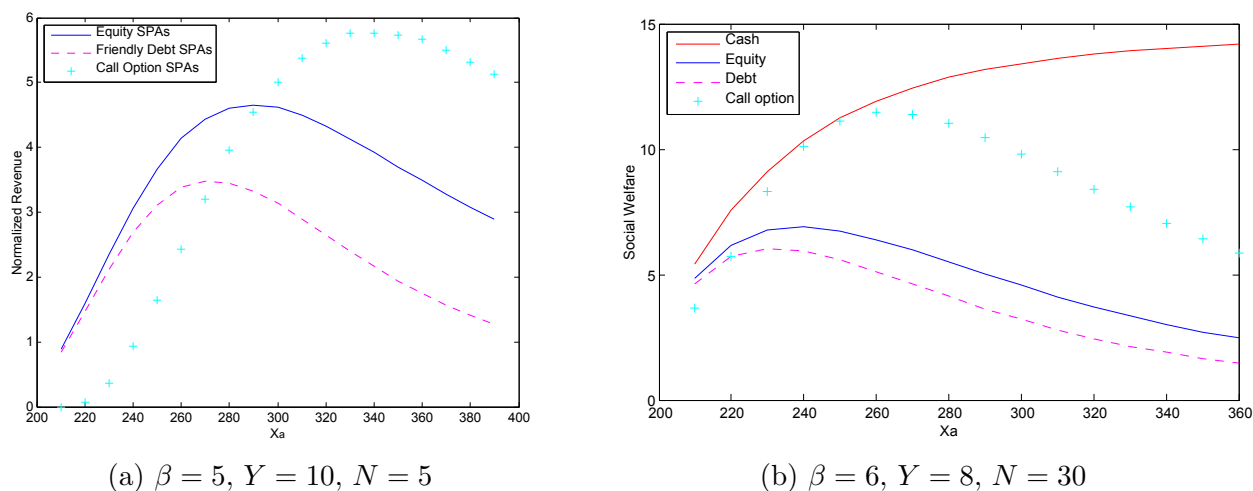


Figure 4: Plots of expected seller’s revenue and social welfare against the auction threshold. With SPAs with equities, friendly debts as defined in Appendix A.7, and call options. One million simulations for θ uniformly distributed in $[200, 500]$, $X_0 = 210$. For simplicity, β is specified instead of the primitives r, μ , and σ .

Auction timing therefore continues to matter with security bids, as seen in Figure 4(a). It also affects the ranking of security designs: among several pure contingent securities, equity gives the highest expected revenue and call option the lowest at $X_a = 280$, whereas call option is the highest and debt is the lowest at $X_a = 360$. The worst security design at $X_a = 300$ more than doubles the revenue from the best security design at $P_a = 220$. Welfare is similarly affected (Figure 4(b)). In this regard, strategic timing could be as important as security design.

²⁰Cong (2017) also derives the optimal security design which entails a bid-specific royalty rate to incentivize the winning bidder to exercise at $X^*(z(\theta_{(1)}))$, thus making the bidder face the same real option as the seller. The optimal design complements this royalty rate with a corresponding bonus payment to ensure truth-telling in the bidding stage. Even though the seller’s and bidders’ preferences for option exercise are now aligned, we note that the “virtual strike price” $z(\theta)$ the seller faces when timing the auction still means the auction occurs later than a constrained-efficient formal auction.

Now for informal auctions, the seller commits to neither a pre-specified timing of the auction nor a bidding and allocation rule. She holds the auction at the most opportune time, and then chooses the bid that gives her the highest expected payoff based on her beliefs regarding the type of each bidder at the time the auction is held.

Cong (2017) shows that in informal auctions, investments are always efficient conditional on auction timing when the seller cannot commit to pre-specified security design, and in equilibrium, every bid is equivalent to cash. The intuition is that the least information-sensitive bids allow a better type to separate from worse types in the cheapest way: to outbid an opponent by one dollar costs the same for all types, but to outbid an opponent by one percent of equity costs a better type strictly more. Moreover, cash bids create the biggest social surplus. Therefore, in equilibrium everyone bids cash (Proposition 5 in Cong (2017)). As such, Lemma 1, and Propositions 2 and 3 apply without modification. The seller times the auctions inefficiently late as long as b is not too big. Moreover, bidders always initiate informal auction, and when $b = 0$ and initiation requires seller approval, bidder-initiation achieves the first-best social welfare.

5.4 Other Specifications

Finally, I briefly remark on how the main economic insights apply to alternative settings.

General Form of Private Information

The driver for the wedge between the seller's option and bidders' option is the information rent the seller has to pay, which in turn depends on the private information bidders have. In general, the asymmetric information could be related to the stochastic process X_t , or have some interactions with it. For example, the results all go through when we allow exercise payoffs of the form $\theta_i X_t - \theta_o$, where θ_i is bidder i 's private information, and θ_o is equal and commonly known across all agents.

From the proofs of Lemma 1, Proposition 2 and Proposition 5, as long as the bidder's payoff as a function of his type is differentiable everywhere, the information asymmetry would lead to the seller's facing real options with increased strike price or reduced cash flow, and the economic intuition goes through. That said, if the private type involves beliefs on the stochastic process such as its drift, we may not obtain closed-form solutions. Moreover,

higher dimensions of informational asymmetry would also complicate the analysis. These constitute interesting future work.

Entrepreneur's Share Retention

As often observed in real life, instead of selling the entire project, the entrepreneur may retain certain shares of the company even after ceding control. How does share retention affect the timing of sale? Let us examine the simple case that the entrepreneur retains a constant α shares of the project.

Share retention does not qualitatively change the results. The seller's private payoff upon option exercise simply becomes $\alpha X_\tau + b$. Normalizing the payoff by α , we can re-label $b' = \frac{b}{\alpha}$. Because of risk-neutrality and the absence of ex-post transfer, the main tradeoffs present in auction timing still remain, though the solution changes quantitatively.

From a social planner's perspective, share retention can potentially mitigate inefficient auction timing. To see this, note that Proposition 2 implies the auction timing is only inefficient for negative or small positive values of b . Requiring the seller to retain a large enough share would make her align more with the social optimal. However, retaining too much reduces the profit of the winning bidder who has to pay the private cost θ , risking inefficiently delaying option exercise further.

Abandonment Option

Is our result driven by the fact that the entrepreneur is selling an investment option? In corporate finance, abandonment options are as prevalent as investment options (e.g., Leland, 1994; Grenadier, Malenko, and Strebulaev, 2014)). Suppose the seller owns an abandonment option: because of the information rent she pays, holding the auction late (at a lower cash flow level) still allows her to postpone the option exercise. The only difference now is that when bidders can initiate, the monotone initiation strategy is increasing in cash flow, i.e., better types initiate earlier at higher cash flow because they recover more upon abandonment.

6 Conclusion

Auctions of real options are ubiquitous, involve tremendous financial resources, and have policy implications. Because the irreversible nature of time endows a seller potential control over the post-auction option exercise, and bidders use their private information when timing an auction, endogenous auction timing and initiation matter. I find that sellers tend to inefficiently delay auctions to increase her expected revenue, leading to delays in option exercises. When bidders can initiate, they always do so in equilibrium. Real options are exercised immediately after bidder-initiated auctions, and the social welfare can be higher. In particular, when bidders and the seller has to jointly agree on bidder initiation of auctions, full social efficiency attains. Taken together, the results complement earlier approaches that either treat auction initiation and timing as exogenous or ignore how post-sale actions affect assets' value, and are especially relevant for sales of assets embedded with real options.

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Appendix: Definitions, Derivations, and Proofs

A.1 Proof of Lemma 1

Proof. FPAs and SPAs are incentive compatible mechanisms. Hence from Proposition 5.2 in Krishna (2009) (also derived as a special case of the proof in Appendix A.8), the expected payment of agent i to the seller is

$$\hat{W} + Q(W_i, W_{-i})W_i - \int_{W=\hat{W}}^{W_i} Q(W, W_{-i})dW = \hat{W} + Q(\theta_i, \theta_{-i})W_i - \int_{\theta=\theta_i}^{\hat{\theta}} Q(\theta, \theta_{-i})D(X_a; X^*(\theta))d\theta$$

The seller's expected revenue from N symmetric bidders is therefore

$$\begin{aligned} & N\mathbb{E}_{\bar{\theta}} \left[\hat{W} + Q(\theta_i, \theta_{-i})W_i - \int_{\theta=\theta_i}^{\hat{\theta}} Q(\theta, \theta_{-i})D(X_a; X^*(\theta))d\theta \right] \\ &= N\mathbb{E}_{\bar{\theta}}[\hat{W}] + N\mathbb{E}_{\bar{\theta}}[Q(\theta_i, \theta_{-i})W_i] - N\mathbb{E}_{\theta_{-i}} \left[\int_{\theta_i=\underline{\theta}}^{\hat{\theta}} f(\theta_i) \int_{\theta=\theta_i}^{\hat{\theta}} Q(\theta, \theta_{-i})D(X_a; X^*(\theta))d\theta d\theta_i \right] \\ &= N\mathbb{E}_{\bar{\theta}}[\hat{W}] + N\mathbb{E}_{\bar{\theta}}[Q(\theta_i, \theta_{-i})W_i] - N\mathbb{E}_{\theta_{-i}} \left[\int_{\theta=\underline{\theta}}^{\hat{\theta}} Q(\theta, \theta_{-i})D(X_a; X^*(\theta)) \int_{\theta_i=\underline{\theta}}^{\theta} f(\theta_i)d\theta_i d\theta \right] \\ &= N\mathbb{E}_{\bar{\theta}}[\hat{W}] + N\mathbb{E}_{\bar{\theta}}[Q(\theta_i, \theta_{-i})W_i] - N\mathbb{E}_{\theta_{-i}} \left[\int_{\theta=\underline{\theta}}^{\hat{\theta}} Q(\theta, \theta_{-i})D(X_a; X^*(\theta)) \frac{F(\theta)}{f(\theta)} f(\theta)d\theta \right] \\ &= N\mathbb{E}_{\bar{\theta}}[\hat{W}] + N\mathbb{E}_{\bar{\theta}} \left[Q(\theta_i, \theta_{-i}) \left(W_i - D(X_a; X^*(\theta_i)) \frac{F(\theta_i)}{f(\theta_i)} \right) \right] \\ &= N\mathbb{E}_{\bar{\theta}}[\hat{W}] + N\mathbb{E}_{\bar{\theta}}[Q(\theta_i, \theta_{-i}) (D(X_a; X^*(\theta_i))[X^*(\theta_i) - z(\theta_i)] - K)], \end{aligned}$$

where the third line follows by changing the order of integration and the fifth line follows from relabeling θ by θ_i .

From individual rationality of participation, the payoff to the cutoff type ($-\hat{W}$) is non-negative, which the seller optimally sets to zero. To maximize revenue, the seller also allocates the project to the type with the highest $D(X_a; X^*(\theta))[X_a^*(\theta) - z(\theta)] - K$. Given the regularity of W_i distribution, the seller optimally allocates the project to $\theta_{(1)}$. We therefore have the lemma. \square

A.2 Proof of Lemma 2

Proof. The regularity condition implies $D(X_a; X^*(\theta))[X_a^*(\theta) - z(\theta)]$ is monotone in θ , therefore it crosses K at most once. Moreover, if $D(X_a; X^*(\theta))[X_a^*(\theta) - z(\theta)] - K \geq 0$, a bidder of type θ definitely wants to participate. Therefore, the cutoff is either that crossing point or $\bar{\theta}$. \square

A.3 Proof of Proposition 1 and Corollary 1.

Proof. Suppose $\hat{\theta} < \bar{\theta}$, then the seller must have set a reserve price that excludes a measurable set of bidder types. Consequently, $D(X_a; X^*(\hat{\theta}))[X_a^*(\hat{\theta}) - z(\hat{\theta})] = K$. This implies either (A) $X_a < X^*(\hat{\theta})$ and $D(X_a; X^*(\hat{\theta}))[X^*(\hat{\theta}) - z(\hat{\theta})] = K$, OR (B) $X_a \geq X^*(\hat{\theta})$ and $X_a = \hat{\theta} + \frac{F(\hat{\theta})}{f(\hat{\theta})}$. As shown in the proof of Lemma 2, the regularity condition implies $D(X_a; X^*(\hat{\theta}))[X_a^*(\hat{\theta}) - z(\hat{\theta})] = K$ has a unique solution. In both cases, the solution is strictly increasing in X_a .

For Corollary 1, the reservation price is $D(X_a; X^*(\hat{\theta}))[X^*(\hat{\theta}) - \hat{\theta}] - K$. In Case B, this is $\frac{F(\hat{\theta})}{f(\hat{\theta})}$, which is increasing in $\hat{\theta}$. Because $\hat{\theta}$ is increasing in X_a in Case B, so R is increasing in X_a . In Case A, the reservation price is $\left(\frac{X_a}{X^*(\hat{\theta})}\right)^\beta \frac{F(\hat{\theta})}{f(\hat{\theta})} = D(X_a; X^*(\hat{\theta}))[X^*(\hat{\theta}) - \hat{\theta}] - K$ is strictly increasing in X_a . Therefore in both cases the reservation price is strictly increasing in X_a . \square

A.4 Proof of Proposition 2

Proof. Given Lemma 1, no matter when the seller holds the auction, there is a corresponding X_a and thus optimal auction reserve price (if allowing reserve price). The seller's expected revenue for holding auction when X_a is first reached can be written as

$$ND(X_0; X_a) \int_{\underline{\theta}}^{\hat{\theta}} d\theta f(\theta) [1 - F(\theta)]^{N-1} \left[D(X_a; X^*(\theta)) \left[X^*(\theta) - \theta - \frac{F(\theta)}{f(\theta)} \right] - K \right]. \quad (13)$$

where $\hat{\theta}$ potentially depends on X_a . The derivative w.r.t. X_a is

$$N \frac{D(X_0; X_a)}{X_a} \int_{\underline{\theta}}^{\hat{\theta}} d\theta f(\theta) [1 - F(\theta)]^{N-1} [\beta K + \mathbf{I}_{\{X_a > X^*(\theta)\}} [\beta z(\theta) - (\beta - 1)X_a]], \quad (14)$$

where I have used the Leibniz formula and the fact that marginal revenue from an interior cutoff type is zero. This expression is continuous in X_a with derivative positive for $X_a \leq \underline{X} \equiv \frac{\beta}{\beta-1}\underline{\theta}$ and negative for $X_a \geq \bar{X} \equiv \frac{\beta}{\beta-1}(K + z(\bar{\theta}))$. Thus there exists X_{opt} in the compact region $[\underline{X}, \bar{X}]$ that maximizes (13). This proves the existence of optimal threshold strategy for auction timing, and the fact that the seller never holds the auction when no bidder would exercise immediately.

Now apply the above argument to welfare. A social planner does not set a reserve price, and the derivative of social surplus w.r.t. X_a is

$$N \frac{D(X_0; X_a)}{X_a} \int_{\underline{\theta}}^{\hat{\theta}} d\theta f(\theta) [1 - F(\theta)]^{N-1} [\beta K + \mathbf{I}_{\{X_a > X^*(\theta)\}} [\beta \theta - (\beta - 1)X_a]], \quad (15)$$

which is weakly smaller than (14) for every X_a . As argued above, an efficient threshold strategy, X_{eff} , also exists. For $K > 0$, (15) is positive at \underline{X} and continuous in $[\underline{X}, \bar{X}]$, which implies that it must be zero at X_{eff} . This in turn implies that $\mathbf{I}_{\{X_a > X^*(\theta)\}} \neq 0$ for some θ , which in turn implies

that (14) is positive at this point X_{eff} (seller's revenue still increasing), ruling out $X_{opt} = X_{eff}$. Meanwhile, it cannot be $X_{eff} > X_{opt}$, for otherwise X_{eff} being welfare-maximizing must imply that integrating (15) over $[X_{opt}, X_{eff}]$ is weakly positive, which in turn implies that integrating (14) over the same region is positive, contradicting X_{opt} being optimal. Therefore, we conclude that $X_{eff} < X_{opt}$ as long as $K > 0$.

Now if $K = 0$, for some distributions, (14) is not guaranteed to be positive. Therefore it is possible X_{opt} can be any $X_a \leq \underline{X}$, which is also socially optimal. Obviously X_{eff} can also be any $X_a \leq \underline{X}$. We thus cannot directly compare X_{opt} and X_{eff} . That said, there exist distributions such that $X_{opt} > \underline{X}$, in which case the optimal auction is delayed relative to a socially efficient one. In conclusion, even without K , the information asymmetry alone can create divergence in preference for auction timing. \square

A.5 Proof of Proposition 3 and Corollaries 3-5

Proof. Conjecture that in a symmetric equilibrium a bidder of type θ initiates the auction with threshold $X_I(\theta)$ that is increasing, and the seller initiates with a threshold X_S . Let $\theta(X_a) = \sup\{\theta : X_I(\theta) \leq X_a\}$. Because the seller cannot commit to an auction rule other than soliciting bids once an auction is initiated, she essentially conducts a second price auction without reserve prices. We denote $\hat{\theta}$ as the cutoff type of participation, which could be constant $\bar{\theta}$ or determined by bidders' IR constraint, in which case the marginal type that bids after initiation must have a valuation of zero.

The time-zero expected payoff to the bidder of type θ following an initiation threshold $X_a \leq X_S$ is

$$\int_{\underline{\theta}}^{\theta(X_a)} d\theta' (N-1) f(\theta') [1 - F(\theta')]^{N-2} \left(\frac{X_0}{X_I(\theta')} \right)^\beta [[W(X_I(\theta'); \theta) - [W(X_I(\theta'); \theta')]^+]^+ + \int_{\theta(X_a)}^{\bar{\theta}} d\theta' (N-1) f(\theta') [1 - F(\theta')]^{N-2} \left(\frac{X_0}{X_a} \right)^\beta [W(X_a; \theta) - [W(X_a; \theta')]^+]^+, \quad (16)$$

where θ' is the first-order statistics of the remaining $N - 1$ bidders. The first term represents the expected payoff when another type θ' initiates before the seller in equilibrium, while the second term represents the expected payoff when this bidder initiates at X_a before other bidders or the seller.

Similarly, the payoff when $X_a > X_S$ is

$$\int_{\underline{\theta}}^{\theta(X_S)} d\theta' (N-1) f(\theta') [1 - F(\theta')]^{N-2} \left(\frac{X_0}{X_I(\theta')} \right)^\beta [[W(X_I(\theta'); \theta) - [W(X_I(\theta'); \theta')]^+]^+ + \int_{\theta(X_S)}^{\bar{\theta}} d\theta' (N-1) f(\theta') [1 - F(\theta')]^{N-2} \left(\frac{X_0}{X_S} \right)^\beta [[W(X_S; \theta) - [W(X_S; \theta')]^+]^+ \quad (17)$$

When $X_a \leq X_S$, $\left(\frac{X_0}{X_a}\right)^\beta [[W(X_a; \theta) - [W(X_a; \theta')^+]^+]^+$, if positive, decreases w.r.t. X_a when $X_a > \frac{\beta}{\beta-1}(\theta + K)$, and increases when $X_a < \frac{\beta}{\beta-1}\theta$. Therefore, differentiating (16) w.r.t. X_a and applying Leibniz's formula gives that in equilibrium $X_I(\theta) \in \left[\max\{X_0, \frac{\beta}{\beta-1}\theta\}, \max\{X_0, \frac{\beta}{\beta-1}(\theta + K)\}\right]$.

Now for the seller, if she uses threshold X_a , the expected payoff is,

$$\begin{aligned} & \int_{\underline{\theta}}^{\theta(X_a)} d\theta' \int_{\underline{\theta}}^{\theta'} d\theta \frac{N(N-1)}{2} f(\theta)f(\theta')[1 - F(\theta')]^{N-2} \left(\frac{X_0}{X_I(\theta)}\right)^\beta [W(X_I(\theta); \theta')]^+ \\ & + \int_{\theta(X_a)}^{\bar{\theta}} d\theta' \int_{\underline{\theta}}^{\theta'} d\theta \frac{N(N-1)}{2} f(\theta)f(\theta')[1 - F(\theta')]^{N-2} \left(\frac{X_0}{X_a}\right)^\beta [W(X_a; \theta')]^+. \end{aligned} \quad (18)$$

Note θ in the integrand is the first-order statistic from the seller's perspective, and therefore initiates before other bidders. Suppose $X_a < X_I(\bar{\theta})$. For any $\theta_i > \theta(X_a)$, the earlier argument leads to $X_a < X^*(\theta_i + K)$, because otherwise i would initiate earlier than X_a which is a contradiction. This implies $X_a < X^*(\theta_i + K)$. Applying the Leibniz formula again, the derivative of (18) is then weakly positive path-by-path because $\left(\frac{X_0}{X_a}\right)^\beta [W(X_a; \theta')]$ is increasing in X_a , the marginal bidder has valuation zero, and the first term does not depend on X_a in the integrand. This implies the integrand is weakly positive path-by-path for any $X_S = X_a < X_I(\bar{\theta})$. Thus the seller optimally initiates at $X_S \geq X_I(\bar{\theta})$, which means almost surely she never initiates in equilibrium.

Now if the bidders initiate, their problem is reduced to expression (16). The derivative at X_a has the same sign as

$$\int_{\theta(X_a)}^{\bar{\theta}} d\theta' f(\theta')[1 - F(\theta')]^{N-2} \frac{d}{dX} \left. \frac{[W(X; \theta) - [W(X; \theta')^+]^+}{X^\beta} \right|_{X=X_a}, \quad (19)$$

which is positive at $\hat{X}(\theta)$ and non-positive at $X^*(\theta + K)$. The integrand is weakly monotone in X_a path-by-path, thus (19) changes sign at a unique $X_a = X_I(\theta)$.

Given (16) is concave in X_a with non-negative cross-partial in X_a and θ , and there exists unique maximizer $X_I(\theta)$, Implicit Function Theorem gives that $X_I(\theta)$ is indeed non-decreasing. A similar argument would rule out a decreasing equilibrium in which the initiator always loses. This ensures (19) is continuous, establishing the optimality of X_I and the validity of an FOC approach. There could be multiple equilibria with different initiation thresholds below X_0 , but in terms of initiation outcome and payoffs, they are all equivalent, making the proposed equilibrium essentially unique.

It should be apparent that when the socially optimal bidder initiation is at $\max\{X_0, \frac{\beta}{\beta-1}(\theta + K)\}$, the initiation is accelerated. Moreover, he would invest in the project right away because once K is sunk, $X^*(\theta) = X_a$ implies it is optimal to exercise the option. Overall, the exercise of the real option is faster than in auctions where only the seller can initiate and the realized winning type might still wait after the auction.

Corollaries 3 and 4 follow directly. Taking the limit of $K \rightarrow 0$, a bidder of type θ initiates the

auction and exercises the option at $\max\{X_0, \frac{\beta}{\beta-1}\theta\}$, which achieves the first-best social efficiency. As K goes to zero, the social efficiency approaches the first-best. However, this is not the case for auctions solely timed by the seller. No matter what auction timing is chosen, it would lead to inefficient outcomes for some realizations of bidder types. Bidder initiation therefore takes advantage of bidders' private information to improve social efficiency. Corollary 5 follows. \square

A.6 Proofs for Proposition 4 and Corollary 6

Proof. The proof on why $X_I(\theta) = X^*(\theta+K)$ is presented in the main text leading to the proposition. Once the bidder promises to pay, the seller has no incentive to delay further, and therefore approves. Note that once the winning bidder pays down K , it is optimal to exercise immediately because $X^*(\theta + K) > X^*(\theta)$.

As for the corollary, a social planner knowing the realizations of all θ_i s would also allocate the real option to $\theta_{(1)}$ at $X^*(\theta + K)$ and immediately exercise the real option. Therefore the outcomes are first-best in terms of social welfare. \square

A.7 Auctions with Security Bids

In practice, we routinely observe competing bids in combinations of cash and contingent securities. Prima facie, the type of bids should not matter as a cash equivalent always exists. One advantage to contingent bids is that they enhance the seller's revenue by effectively linking pay-off to a variable affiliated with bidders' private information—the “linkage” principle in Milgrom (1985). Contingent bids also mitigate liquidity or legal constraints and reduce valuations gaps among various parties. My definition of security bids is motivated by these observations.

Security bids. If the contingent securities are entirely written on $X_\tau - \theta$ when the option is exercised at $t = \tau$, they do not distort the winning bidder's incentives for option exercise and are equivalent to cash bids. However if the security is written on X , it introduces post-auction moral hazard as in Cong (2017): due to the non-contractibility of θ , the winning bidder bears the private cost, but only claims partial cash flow from the exercise. We now extend our analysis to this setting.

The inefficient delay depends on the specific security used because the latter affects auction timing through τ_1^* , yet the main tradeoff remains. As long as the seller's virtual valuation differs from true option value, timing auctions leads to substantial variations in revenues, potentially at the expense of welfare.

I follow Cong (2017) to define ordered standard security bids as follows.

DEFINITION. A *standard security bid* is an upfront cash payment $C \in \mathbb{R}$ and a contingent payment at the time of investment τ given by continuous function $S(X_\tau) \in \mathbb{R}$.

An *ordered set of securities* ranked by index s is defined by a left-continuous map $\Pi(s) = \{C(s), S(s, \cdot)\}$ from $[s_L, s_H] \subset \mathbb{R}$ to the set of standard security bids such that for each voluntary participant of type θ , $V(s, \theta) \equiv \tilde{V}(C(s), S(s, \cdot), \theta)$ is non-negative and non-increasing in s on $[s_L, \tilde{s}]$ and negative on $(\tilde{s}, s_H]$ for some $\tilde{s} \in [s_L, s_H]$.

Standard security bids are simple and intuitive and, as shown in Cong (2017), can implement the optimal auction design in an extended space of securities. In addition to being standard, an ordered set of securities admits one-dimensional ranking with index s for any payoff from the project, and permissible bids cover a wide range such that each participant earns a non-negative profit by bidding low enough but earns no profit by bidding too high. Any such sets can be represented by the mapping defined above up to an order-preserving transformation of the index. s could be the fraction of shares α in a pure equity auction $\{C(\alpha) = 0, S(\alpha, X) = \alpha X\}$, the (negative) strike price k in a call-option auction $\{C(-k) = 0, S(-k, X) = \max\{X - k, 0\}\}$, or the bonus B in a bonus-bid auction with royalty rate ϕ fixed $\{C(B) = B, S(B, X) = \phi X\}$. M&As, VC contracts, and lease auctions routinely use such securities, and indeed the bidder offering the highest s wins.²¹

The numerical illustrations to follow sometimes include another common form of security: a fixed promise of payment B from the project's payoff—essentially debts without interests, $S(B, X) = \min(X, B)$, also known as friendly debt, or in Islamic finance, Qard/Qardul hassan.²²

Now define

$$\tilde{V}(C, S(\cdot), \theta) = \max_{\tau \geq t_a} \mathbb{E}_X[e^{-r(\tau-t_a)}(X_\tau - S(X_\tau) - \theta)] - C - K, \quad (20)$$

where τ is any stopping time. $S(X)$, being of general form, distinguishes this problem from traditional real-options models. Cong (2017) shows that under mild regularity conditions, a threshold investment strategy exists that is optimal among all stopping times. Moreover, the valuation $\tilde{V}(C, S(\cdot), \theta)$ is continuously decreasing in θ .

Informal auctions with security bids. Because auction timing, bidding, and investment involve sequential actions, the equilibrium concept for informal auctions is *Perfect Bayesian Equilibrium*. Informal auctions therefore exhibit signaling through security choices. In particular, it is

²¹In M&As with the acquirer's stocks as bids, C simply corresponds to the value of the acquirer's cash flows that are independent of the acquisition, and X is the payoff from the acquired assets and projects, and the synergy created. Cong (2017) discusses how the seller may want to restrict the range of allowable bids.

²²Interestless debts are used frequently in contractual agreements in Islamic banking and microfinance, and are equivalent to granting the winning bidder instead of the seller call options - the exact opposite situation to that for call option bids.

the most natural to formally define an English auction with security bids:

1. The seller initiates the auction at some time t_a and all agents enter the bidding stage.
2. The seller gradually increases a numerical score R from $R = Y$, and a bidder remains in the auction if he can deliver an informal bid from a "feasible set" $\{\Pi : R(\Pi) \geq R\}$. The auction ends when only one bidder is left, and he chooses an informal bid from the final "feasible set."
3. The winning bidder i pays the upfront cash C^i and the initial cost X at t_a , and then invests rationally at $\tau_{\theta_i}^i$ and makes the contingent payment $S^i(P_{\tau_{\theta_i}^i})$, where C^i and S^i are given by his chosen final bid.

Note this variant of the English auction is equivalent to SPAs, in which bidders bid a score they generate and the winner pays the second-highest score bid.²³ This a priori is different from SPAs in which the winning bidder pays the informal bid corresponding to the second highest score. The distinction is important because the same security bids generally cost the buyers differently.

A.8 Proof for Proposition 5

Proof. Again, I use the Direct Revelation Principle and focus on a truth-telling mechanism. Let $Q(\tilde{\theta}_i, \theta_{-i})$ be the probability of allocating the project to bidder i who reports $\tilde{\theta}_i$, who has investment cost $\Theta(\theta_i, \theta_{-i})$. In the baseline model, $\Theta(\theta_i, \theta_{-i}) = \theta_i$.

The expected revenue at time zero to type θ_i upon participating and optimally investing is

$$U(\theta_i, \tilde{\theta}_i) = \mathbb{E}_{\theta_{-i}} \left[Q(\tilde{\theta}_i, \theta_{-i}) \max_{\tau \geq t_a} \mathbb{E}_X \left[e^{-r\tau} (X_\tau - \Theta(\theta_i, \theta_{-i})) - \int_{t_a}^{\infty} e^{-rt} S(\tilde{\theta}_i, \theta_{-i}, \mathcal{I}_t) dt - e^{-rt_a} (Y + K) \right] \right],$$

where $S(\tilde{\theta}_i, \theta_{-i}, \mathcal{I}_t)$ is a general security payment (including cash) that potentially depends on everyone's report and the common information filtration for both the seller and winning bidder. In the baseline model, $S(\tilde{\theta}_i, \theta_{-i}, \mathcal{I}_t) = rB(\tilde{\theta}_i)$, which makes the integral term the cash bid $B(\tilde{\theta}_i)$ at the time of auction. As $S(\tilde{\theta}_i, \theta_{-i}, \mathcal{I}_t)$ could be artificially constructed such that an optimal stopping time for exercising the real option may not exist, it is reasonable to focus attention on the set of $S(\tilde{\theta}_i, \theta_{-i}, \mathcal{I}_t)$ such that an optimal stopping time exists for all types under a direct mechanism. Cong (2017) provides a sufficient condition.

With this restriction, let $\tau^*(\theta_i, \tilde{\theta}_i, \theta_{-i})$ denote the optimal stopping time that is almost surely bigger than t_a , and $\tau_i^* = \tau^*(\theta_i, \theta_i, \theta_{-i})$. Incentive compatibility requires $U(\theta_i) \equiv U(\theta_i, \theta_i) \geq U(\theta_i, \tilde{\theta}_i)$ and the individual rationality requires $U(\theta_i) \geq 0$.

²³Defining an ascending auction with multiple security bids is challenging. See also Gorbenko and Malenko (2017), one of the first studies to formalize an English auction with security bids (both cash and equity).

The IC constraint can be written as $\theta_i \in \operatorname{argmax}_{\tilde{\theta}_i \in [\underline{\theta}, \bar{\theta}]} U(\theta_i, \tilde{\theta}_i) \forall i$. Let $a = (\tau, \tilde{\theta})$ denote the action pair of reporting $\tilde{\theta}$ and rationally exercise following the stopping time τ . Let

$$g(a, \theta) = Q(\tilde{\theta}, \theta_{-i}) \mathbb{E}_X \left[e^{-r\tau} (X_\tau - \Theta(\theta_i, \theta_{-i})) - \int_{t_a}^{\infty} e^{-rt} S(\tilde{\theta}, \theta_{-i}, \mathcal{I}_t) dt - e^{-rt_a} (Y + K) \right]$$

Then following the argument in Milgrom and Segal (2002), for any $\theta', \theta'' \in [\underline{\theta}, \bar{\theta}]$ with $\theta' < \theta''$,

$$\begin{aligned} |U(\theta') - U(\theta'')| &= \mathbb{E}_{\theta_{-i}} \left[\left| \sup_{a'} g(a', \theta') - \sup_{a''} g(a'', \theta'') \right| \right] \\ &\leq \mathbb{E}_{\theta_{-i}} \left[\sup_a |g(a, \theta') - g(a, \theta'')| \right] = \mathbb{E}_{\theta_{-i}} \left[\sup_a \left| \int_{\theta''}^{\theta'} g_\theta(a, \theta) d\theta \right| \right] \\ &\leq \mathbb{E}_{\theta_{-i}} \left[\int_{\theta'}^{\theta''} \sup_a |g_\theta(a, \theta)| d\theta \right] \leq A |\theta'' - \theta'| \end{aligned}$$

This implies $U(\theta)$ is absolutely continuous, and thus differentiable everywhere. $U(\theta) = U(\hat{\theta}) - \int_{\hat{\theta}}^{\theta} U'(\theta') d\theta'$. By Theorem 1 in Milgrom and Segal (2002), $U'(\theta) = g_\theta(a^*, \theta)$. Writing it in the integral form gives that any incentive compatible and individually rational mechanism satisfies

$$U(\theta_i) = \mathbb{E}_{\theta_{-i}} \left[\int_{\theta_i}^{\hat{\theta}} Q(\theta_j, \theta_{-i}) \mathbb{E}_X [e^{-r\tau_j^*}] \Theta_1(\theta_j, \theta_{-i}) d\theta_j \right] + U(\hat{\theta}) \quad (21)$$

where $U(\hat{\theta}) \geq 0$. Moreover $\tau_i \geq t_a, \forall i$ for time consistency.

The ex-ante social welfare is $N \mathbb{E}_{\tilde{\theta}} [Q(\theta_i, \theta_{-i}) (\mathbb{E}_X [e^{-r\tau_i^*} (X_{\tau_i^*} - \Theta(\theta_i, \theta_{-i}) + b)] - e^{-rt_a} (Y + K))]$, and the seller's *ex ante* revenue is the social welfare less the agents' ex-ante utilities:

$$N \mathbb{E}_{\tilde{\theta}} [Q(\theta_i, \theta_{-i}) (\mathbb{E}_X [e^{-r\tau_i^*} (X_{\tau_i^*} - \Theta(\theta_i, \theta_{-i}) + b)] - e^{-rt_a} (Y + K))] - N \mathbb{E}_{\tilde{\theta}} [U(\theta_i)].$$

Using (21) and taking expectations over the winning bidder's type, it becomes

$$N \mathbb{E}_{\tilde{\theta}} [Q(\theta_i, \theta_{-i}) (\mathbb{E}_X [e^{-r\tau_i^*} (X_{\tau_i^*} - \Theta(\theta_i, \theta_{-i}) - F(\theta_i)/f(\theta_i) \Theta_1(\theta_i) + b)] - e^{-rt_a} (Y + K))] - N U(\hat{\theta})$$

The seller optimally sets $U(\hat{\theta}) = 0$. To maximize revenue, the seller allocates to the best type, if at all. The expression in the proposition follows.

The optimal cutoff type is set to contribute zero revenue upon winning, therefore the marginal bidder i has a type θ solving

$$D(X_a; X^*(\theta)) \left[X^*(\theta) - \mathbb{E}_{-i} [\Theta(\theta, \theta_{-i})] - \frac{F(\theta)}{f(\theta)} \Theta_1(\theta) + b \right] - Y - K = 0 \quad (22)$$

The regularity condition requires the LHS to be monotone in θ , yielding a unique cutoff.

With cash bids, $X_{\tau^*} = X^*(\theta_{(1)})$, but for other security-bid FPAs or SPAs, X_{τ^*} may depend on

$\theta_{(2)}$ in SPAs. Denote $X_{\tau^*(\theta_{(1)}, \theta_{(2)})}$ as X_{τ^*} . The seller's expected utility for holding auction when X_a is first reached can be written as

$$D(X_0; X_a) \int_{\underline{\theta}}^{\hat{\theta}} d\theta \int_{\theta}^{\bar{\theta}} d\theta' \frac{N(N-1)f(\theta)f(\theta')}{2[1-F(\theta')]^{2-N}} \left[D(X_a; X_{\tau^*}) \left[X_{\tau^*} - \theta - \chi \frac{F(\theta)}{f(\theta)} \Theta_1(\theta) + b \right] - Y - K \right]. \quad (23)$$

where $\chi = 1$, $X_{\tau^*}(\theta)$ is the winning bidder's investment threshold, and $\hat{\theta}$ potentially depends on X_a . Here θ' is the best realized type from the remaining $N - 1$ bidders. The derivative w.r.t. X_a is

$$\frac{D(X_0; X_a)}{X_a} \int_{\underline{\theta}}^{\hat{\theta}} d\theta \int_{\theta}^{\bar{\theta}} d\theta' \frac{N(N-1)f(\theta)f(\theta')}{2[1-F(\theta')]^{2-N}} \left[\beta(Y + K) + \mathbf{I}_{\{X_a > X_{\tau^*}\}} \left[\beta \left(\theta + \frac{F(\theta)}{f(\theta)} \Theta_1(\theta) - b \right) - (\beta - 1)X_a \right] \right], \quad (24)$$

where I have used the Liebzniz formula and the fact that marginal revenue from an interior cutoff type is zero. This expression is continuous in X_a with derivative positive for $X_a \leq \underline{X} \equiv \frac{\beta}{\beta-1}(\underline{\theta} - b)$ and negative for $X_a \geq \bar{X} \equiv \frac{\beta}{\beta-1}(Y + K + \left(\theta + \frac{F(\theta)}{f(\theta)} \Theta_1(\theta) - b\right))$. Thus there exists X_a in the compact region $[\underline{X}, \bar{X}]$ that maximizes (23). This proves the existence of optimal threshold strategy for auction timing, and the fact that the seller never holds the auction when no bidder would exercise immediately. When we have security bids and X_{τ^*} is not necessarily $X^*(\theta_{(1)})$, we can redefine $\underline{X} \equiv \min_{\theta} X_{\tau^*}(\theta)$. \square

A.9 Proof of Corollaries 7 and 8

Proof. Now apply the argument in the proof for Proposition 5 to welfare, an efficient threshold strategy exists, and the derivative of social surplus w.r.t. X_a is

$$\frac{D(X_0; X_a)}{X_a} \int_{\underline{\theta}}^{\hat{\theta}} d\theta \int_{\theta}^{\bar{\theta}} d\theta' \frac{N(N-1)f(\theta)f(\theta')}{2[1-F(\theta')]^{2-N}} \left[\beta(Y + K) + \mathbf{I}_{\{X_a > X_{\tau^*}\}} [\beta(\theta - b) - (\beta - 1)X_a] \right], \quad (25)$$

which is smaller than (24) for every X_a . At the optimal threshold X_{opt} , integrating (24) over $[X_{opt}, X_a]$ must be weakly negative for any $X_a > X_{opt}$. Thus integrating (25) over $[X_{opt}, X_a]$ must be also weakly negative for any $X_a > X_{opt}$, implying the efficient threshold $X_{eff} \leq X_{opt}$. Now at X_{eff} , (25) is necessarily zero, otherwise it is not a local maximum. Because $X_{eff} \geq \underline{X}$, integrating (24) over $[X_{eff}, X_{eff} + \epsilon]$, where $\epsilon > 0$ is infinitesimal, must be positive. This implies $X_{eff} < X_{opt}$. Another way to see this is that Equation (23) with $\chi = 0$ corresponds to welfare, and the equation is supermodular in (X_a, χ) . Thus a seller optimally delays the auction beyond the socially efficient threshold given the same security design and allocation rule. When $b < 0$, the option exercise by the winning bidder could be inefficiently early ($X_a < X^*(\theta_{(1)} - b)$) or late ($X_a > X^*(\theta_{(1)} - b)$); when $b > 0$, it is always inefficiently late.

Now if b is sufficiently large, i.e. $b > Y + K + \bar{\theta} - \underline{\theta}$, even though the seller prefers an early option exercise, the socially optimal auction threshold is lower than $X^*(\underline{\theta})$, making (24) and (25) equal. Therefore the maximizer for the seller's payoff must also be the maximizer for the social planner. The auction is thus efficiently timed, and the option exercise by the winning bidder is inefficiently late because he does not internalize the positive b . \square